Existence of Nash Equilibrium in Competitive Nonlinear Pricing Games with Adverse Selection

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Abstract

We show that for a large class of competitive nonlinear pricing games with adverse selection, the property of better-reply security is naturally satisfied - thus, resolving via a result due to Reny (1999) the issue of existence of Nash equilibrium for a large class of competitive nonlinear pricing games.

1 Introduction

In his paper on the existence of Nash equilibria in discontinuous games, Reny (1999) introduced the notion of *better-reply security*, and showed that any game with compact, convex strategy spaces and payoffs at least quasiconcave in each player's strategies possess Nash equilibria if in addition, the game is better-reply secure. A game is said to be better-reply secure if for every nonequilibrium strategy, x^*, and every payoff vector limit u^*, generated by strategies approaching x^* some player has a strategy yielding a payoff strictly above u_i^* even if other players deviate slightly from x^*. The main contribution of this paper is to show that for a large class of competitive nonlinear pricing games with adverse selection, the property of better-reply security is naturally satisfied - thus, resolving the issue of existence of Nash equilibrium for a large class of competitive nonlinear pricing games.

The question of existence of Nash equilibria for competitive nonlinear pricing games with adverse selection is particularly difficult for two reasons: (1) the complicated nature of each firm's strategy space, and (2) payoff discontinuities. In a nonlinear pricing game, each firm's strategy has two components: a product line and a pricing function (i.e., a mapping from the product line into prices). Thus,

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each firm's strategy space consists of a set of product line-pricing function pairs. In addition, the presence of adverse selection requires that each product line-pricing function pair be incentive compatible. Our approach to the Nash equilibrium existence question consists of three steps: First, we construct the product-price catalog game strategically equivalent to the nonlinear pricing game. Second, we prove existence of a Nash equilibrium for the mixed extension of the catalog game via Reny's (1999) result by showing that the catalog game is better-reply secure. Third, invoking the competitive taxation principle (Page (1999) and Page and Monteiro (2003)) which establishes the equivalence of catalog games and nonlinear pricing games, we deduce existence of Nash equilibrium for the mixed extension of the nonlinear pricing game from existence of Nash equilibrium in the mixed catalog game.

By considering the catalog game rather than the nonlinear pricing game we are able to avoid many of the strategy space complications which arise in nonlinear pricing games and to rid the analysis of incentive compatibility constraints altogether. Moreover, by moving to catalog games we are able to show that a large class of nonlinear pricing games possess the property of better-reply security. This fact, in turn, allows us to apply Reny's result to deduce the existence of a Nash equilibrium - and thus to get around the problems caused by discontinuities. In the catalog game, rather than each firm offering the agent an incentive compatible product line-pricing function pair, each firm simply offers the agent a catalog of product-price pairs and delegates choice from the catalog to the agent. By the competitive taxation principle, this move from product line-pricing function pairs to catalogs involves no loss of generality.

We shall proceed as follows: In Section 2, we define the primitives of the competitive nonlinear pricing model we shall consider, define the notion of a catalog, and specify the agent's contracting problem under delegated competitive contracting via catalogs. Also, in Section 2 we deduce the agent's best response mapping and specify each firm's expected profit function. In Section 3, we specify the catalog game and its mixed extension and we define the notions of Nash equilibrium for catalog games and mixed catalog games. Also, in Section 3, we state our main result on the existence of Nash equilibrium for the mixed catalog game. Also, in Section 3, we demonstrate the existence of Nash equilibrium for any mixed catalog game corresponding to our primitives by showing that all such games are better-reply secure.

2 The Catalog Model

2.1 Primitives

2.1.1 Agent Types and Sales Contracts

We shall assume that

(A-1) the set of agent types is given by a probability space, $(T, B(T), \mu)$, where T is a Borel space, $B(T)$ is the Borel σ -field in T, and μ is a probability measure defined on $B(T)$.

Recall that a Borel space is a Borel subset of a complete separable metric space. Under (A-1), multidimensional type descriptions are allowed.

Suppose now that there are m firms indexed by i *and* j ($= 1, 2, \ldots, m$) and let X be a subset of R^L representing the set of products firms can offer the agent and let D be a subset of the real numbers R representing the prices firms might charge. Now let

$$K := X \times D.$$

Elements of K, denoted by (x, p), can be viewed as describing the relevant characteristics of the sales contracts offered by firms. For example, given sales contract $(x, p) \in K$, the vector $x = (x_1, \ldots, x_L) \in X$ describes the product characteristics such as quantity, quality, and location, while $p \in D$ gives the price. For each firm $i = 1, 2, \ldots, m$, let K_i be a subset of K containing all the sales contracts that firm i can offer to the agent. The set K_i, then, is the i^{th} firm's feasible set of products and prices.

We shall assume that,

(A-2) (i) X is closed and bounded and contains the zero vector, (ii) D is closed and bounded and contains zero, and (iii) for each firm $i = 1, 2, \ldots, m$, the feasible set of contracts K_i is a closed subset of K containing $(0, 0)$.

In order to take into account the possibility that the agent may wish to abstain from contracting altogether, we include in our list of feasible contract sets the set K_0 given by

$$K_0 := \{(0, 0)\}. \tag{1}$$

Letting $I = \{0, 1, 2, \ldots, m\}$, define the set

$$\mathbb{K} := \{(i, x, p) \in I \times X \times D : (x, p) \in K_i\}. \tag{2}$$

A firm-contract pair $(i, x, p) \in \mathbb{K}$ indicates that the agent has chosen sales contract $(x, p) \in K_i$ from firm i, while $(i, x, p) = (0, 0, 0) \in \mathbb{K}$ indicates that the agent has chosen to abstain from contracting altogether. Note that the set $\mathbb{K}$ is a closed subset of the compact set $I \times X \times D$. [1] Thus, $\mathbb{K}$ is a compact set.

2.1.2 The Agent's Utility Function

We shall assume that

(A-3) the agent's utility function $v(t, \cdot, \cdot, \cdot) : \mathbb{K} \to \mathbb{R}$ is given by

$$v(t, i, x, p) = u(t, i, x) - p, \tag{3}$$

[1] Equip I with the discrete metric $d_I(\cdot, \cdot)$ given by

$$d_I(i, i') = \begin{cases} 1 & \text{if } i \neq i' \\ 0 & \text{otherwise.} \end{cases}$$

where, (i) for each $t \in T$, $u(t,\cdot,\cdot)$ is continuous in (i,x) and for each $(i,x) \in I \times X$, $u(\cdot,i,x)$ is $B(T)$-measurable, (ii) $u(t,i,0) < u(t,0,0)$ for all $t \in T$ and $i = 1,2,\ldots,m$, and (iii) for each $i \in I$ the family of functions $U_i := \{u(t,i,\cdot) : t \in T\}$ is equicontinuous (i.e., for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $\|x' - x\| < \delta$ then $|u(t,i,x') - u(t,i,x')| < \varepsilon$ for all $t \in T$).

Note that we allow the agent's utility to depend not only on the sales contract (x,p) but also on brand name i (i.e., the name of the firm with which the agent contracts). However, by (A-3)(ii) if the agent is to derive any utility from a firm's brand name beyond the reservation level, $v(t,0,0,0)$, then the agent must enter into a contract with the firm. Allowing utility to depend on brand names *does not* rule out the possibility that some (or all) types of the agent are completely indifferent to brand names. Also, note that assumption (A-3)(iii), equicontinuity, will be satisfied automatically if the set of agent types T is compact and for each $i \in I$, $u(\cdot,i,\cdot)$ is continuous on $T \times X$. Finally, note that given the way we have set up the agent's choice set, $\mathbb{K}$, and the agent's utility function, we are assuming that the agent contracts with one and only one firm.

2.1.3 The Firm's Profit Function

We shall assume that

(A-4) the jth firm's profit function, $\pi_j(\cdot,\cdot,\cdot) : \mathbb{K} \to R$ is given by

$$\pi_j(i,x,p) := (p - c_j(x))\, I_j(i), \tag{4}$$

where

$$I_j(i) = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ otherwise,} \end{cases}$$

and where the cost function $c_j(\cdot) : X \to R_+$ is nonnegative and lower semicontinuous.

Under (A-4) the j^{th} firm's profit function $\pi_j(\cdot,\cdot,\cdot) : \mathbb{K} \to R$ is upper semicontinuous on the compact set $I \times X \times D$. Note that each firm's profit function does not depend directly on the agent's type. Also note that if the agent chooses not to contract with firm j, then firm j's profit is zero.

2.2 Catalog Games

2.2.1 Catalogs

For each firm $i = 1,2,\ldots,m$, let C_i be a nonempty, closed subset of K_i. We can think of the subset C_i as representing a *catalog* of contracts that the i^{th} firm might offer to the agent. For $i = 0,1,2,\ldots,m$, let $P_f(K_i)$ denote the collection of all possible catalogs, that is, the collection of *all nonempty, closed subsets of* K_i.[2] Since

[2] Note that since $K_0 = \{(0,0)\}$, $P_f(K_0)$ consists of a one nonempty, closed subset, namely the set $\{(0,0)\}$.

$K_i \subseteq X \times D$ is a compact subset of $R^L \times R$, the collection of catalogs, $P_f(K_i)$, equipped with the Hausdorff metric h is automatically compact (see Aliprantis and Border (1999) for the definition of the Hausdorff metric and a discussion).

If firms compete via catalogs, then their strategy choices can be summarized via a catalog profile,

$$(C_1, \ldots, C_m) \,. \tag{5}$$

Here, the i^{th} component of the m-tuple $(C_1, \ldots, C_m)$ is the catalog offered by the i^{th} firm to the agent. Let

$$\mathbf{P} := P_f(K_1) \times \cdots \times P_f(K_m)$$

denote the space of all catalog profiles. If $\mathbf{P}$ is equipped with the metric $h_{\mathbf{P}}$ given by

$$h_{\mathbf{P}}\left((C_1, \ldots, C_m), (C'_1, \ldots, C'_m)\right) := \max\{h(C_i, C'_i) : i = 1, 2, \ldots, m\}, \tag{6}$$

then the space of catalog profiles $(\mathbf{P}, h_{\mathbf{P}})$ is a compact metric space.

2.2.2 The Agent's Catalog Choice Problem

Given catalog profile,

$$(C_1, \ldots, C_m) \,,$$

the agent's choice set is given by

$$\Gamma(C_1, \ldots, C_m) := \{(i, x, p) \in \mathbb{K} : (x, p) \in C_i\}, \tag{7}$$

and the agent's choice problem is given by

$$\max\{v(t, i, x, p) : (i, x, p) \in \Gamma(C_1, \ldots, C_m)\} \,. \tag{8}$$

Recall that $K_0 := \{(0, 0)\}$. Thus, $P_f(K_0) = \{(0, 0)\}$ and the agent can choose to abstain from contracting altogether by choosing from $C_0 = \{(0, 0)\}$ - and thus, *participation is endogenously determined.*

Under assumptions (A-1)-(A-3), for each t the agent's choice problem has a solution. Let

$$v^*(t, C_1, \ldots, C_m) := \max\{v(t, i, x, p) : (i, x, p) \in \Gamma(C_1, \ldots, C_m)\} \tag{9}$$

and

$$\Phi(t, C_1, \ldots, C_m) := \{(i, x, p) \in \Gamma(C_1, \ldots, C_m) : v(t, i, x, p) = v^*(t, C_1, \ldots, C_m)\}. \tag{10}$$

The set-valued mapping

$$(C_1, \ldots, C_m) \to \Phi(t, C_1, \ldots, C_m)$$

is a type t agent's best response mapping. For each catalog profile

$$(C_1, \ldots, C_m) \in P_f(K_1) \times \cdots \times P_f(K_m),$$

$\Phi(t, C_1, \ldots, C_m)$ is a nonempty closed subset of $\mathbb{K}$.

The following Proposition summarizes the continuity and measurability properties of the mappings, Γ and Φ, and the optimal utility function, v^*.

Proposition *(Continuity and measurability properties): Suppose assumptions (A-1)-(A-3) hold. Then the following statements are true. (a) The choice correspondence* $\Gamma(\cdot, \ldots, \cdot)$ *is* $h_{\mathbf{P}}$*-continuous on the space of catalog profiles* $\mathbf{P}$ *(i.e., is continuous with respect to the metric* $h_{\mathbf{P}}$*, (b) The function* $v^*(\cdot, \cdot, \ldots, \cdot)$ *is* $h_{\mathbf{P}}$*-continuous on* $\mathbf{P}$ *for each* $t \in T$*, and is* $B(T)$*-measurable on* T *for each* $(C_1, \ldots, C_m) \in \mathbf{P}$*. (c) For each* $t \in T$*,* $\Phi(t, \cdot, \ldots, \cdot)$ *is* $h_{\mathbf{P}}$ *-upper semicontinuous on* $\mathbf{P}$ *and* $\Phi(\cdot, \cdot, \ldots, \cdot)$ *is* $B(T) \times B(\mathbf{P})$ *-measurable on* $T \times \mathbf{P}$*.*[3]

The proof of the Proposition above follows from Propositions 4.1 and 4.2 in Page (1992). It is easy to show that if X and D are finite, then for each $t \in T$, $\Phi(t, \cdot, \ldots, \cdot)$ is $h_{\mathbf{P}}$-continuous on $\mathbf{P}$.

2.2.3 Catalogs and Nonlinear Pricing Schedules

If we model firms as competing via nonlinear pricing schedules, then their strategy choices are given by a profile of nonlinear pricing schedules,

$$((X_1, p_1(\cdot)), \ldots, (X_m, p_m(\cdot))), \tag{11}$$

where X_j is the j^{th} firm's product line and $p_j(\cdot)$ is the j^{th} firm's pricing function (i.e., a mapping from the product line into prices). A close relationship exists between catalogs and nonlinear pricing schedules and this relationship is summarized in the competitive taxation principle (Page (1999) and Page and Monteiro (2003) stated below. We begin with a definition of implementable nonlinear pricing schedules, followed by a discussion of the agent's choice problem under nonlinear pricing schedules..

Definition (Implementable Nonlinear Pricing Schedules)

An implementable nonlinear pricing schedule is a pair, $(X_j, p_j(\cdot))$*, where* X_j *is a nonempty, closed subset of* X *representing the* j^{th} *firm's product line, and* $p_j(\cdot)$ *is a real-valued, lower semicontinuous function, defined on* X_j *taking values in the set of prices* D *such that*

$$graph\{p_j(\cdot)\} := \{(x, p) \in X \times D : p = p_j(x)\} \subseteq K_j.$$

[3]Here $B(\mathbf{P})$ denotes the Borel σ-field in the compact metric space $(\mathbf{P}, h_{\mathbf{P}})$. Moreover,

$$B(\mathbf{P}) = B(P_f(K_1)) \times \cdots \times B(P_f(K_m)),$$

where $B(P_f(K_j))$ denotes the Borel σ-field in the compact metric space $(P_f(K_j), h)$ (see Aliprantis and Border (1999) Theorem 4.43, p. 146).

Agent Choice Under Nonlinear Pricing Schedules In order to take into account the possibility that the agent may wish to abstain from contracting altogether, we include in our list of schedules the schedule,

$$\begin{gathered}(X_0, p_0(\cdot)) \\ \text{where} \\ X_0 := \{0\} \text{ and } p_0(0) := 0.\end{gathered} \tag{12}$$

Given $m+1$-tuple of nonlinear pricing schedules,

$$((X_0, p_0(\cdot)), (X_1, p_1(\cdot)), \ldots, (X_m, p_m(\cdot))),$$

the agent's choice set is given by

$$\Lambda(X_0, X_1, \ldots, X_m) := \{(i, x) \in I \times X : x \in X_i\}, \tag{13}$$

where $I = \{0, 1, 2, \ldots, m\}$.[4] Given choice $(i, x) \in \Lambda(X_0, X_1, \ldots, X_m)$, a type t agent's utility is given by

$$v(t, i, x, p_i(x)) = u(t, i, x) - p_i(x).$$

The Competitive Taxation Principle For the convenience of the reader, we restate below the competitive taxation principle from Page and Monteiro (2003). This principle makes clear the close relationship between catalogs and nonlinear pricing schedules. We begin by recalling the notion of a direct mechanism. Within the context of competitive nonlinear pricing considered here, a direct mechanism,

$$t \to (i(t), x(t), p(t)),$$

is a function defined on the set of agent types specifying for each agent type t the firm $i(t)$, the product $x(t)$, and the price $p(t)$ the mechanism intends the type t agent to choose. Letting

$$\Sigma(C_1, \ldots, C_m)$$

denote the set of all measurable selectors from the best response mapping

$$t \to \Phi(t, C_1, \ldots, C_m),$$

(i.e., the set of all functions $(i(\cdot), x(\cdot), p(\cdot))$ such that $(i(t), x(t), p(t)) \in \Phi(t, C_1, \ldots, C_m)$ for all t), it is possible to show that a direct mechanism $(i(\cdot), x(\cdot), p(\cdot))$ is incentive compatible and individually rational (and therefore to show that the choices intended by the mechanism can be realized) if and only if $(i(\cdot), x(\cdot), p(\cdot)) \in \Sigma(C_1, \ldots, C_m)$ for some catalog profile $(C_1, \ldots, C_m) \in \mathbf{P}$ (see Theorem 1 in Page and Monteiro (2003)). By the Kuratowski-Ryll-Nardzewski Selection Theorem, $\Sigma(C_1, \ldots, C_m)$ is nonempty for any catalog profile $(C_1, \ldots, C_m)$ (see Aliprantis and Border (1999), p. 567).

[4] Again, recall that firms are indexed by i *and* j.

Theorem 1 *(The Competitive Taxation Principle)*

Suppose assumptions (A-1)-(A-4) hold. Then the following statements are true:

1. *For each catalog profile,*
$$(C_1, \ldots, C_m),$$
there exists a unique profile of implementable nonlinear pricing schedules,
$$((X_1, p_1(\cdot)), \ldots, (X_m, p_m(\cdot))),$$
such that for each firm $j \in I$
$$X_j = proj_X C_j$$
$$\text{and}$$
$$p_j(x) = \min\{p \in D : (x,p) \in C_j\} \text{ for all } x \in X_j,$$
and such that for all direct contracting mechanisms,
$$(i(\cdot), x(\cdot), p(\cdot)) \in \Sigma(C_1, \ldots, C_m),$$
and for agent types $t \in T$
$$i(t) \in I,\ x(t) \in X_{i(t)},\ \text{and}$$
$$p(t) = p_{i(t)}(x(t)).$$

2. *For each profile of implementable nonlinear pricing schedules,*
$$((X_1, p_1(\cdot)), \ldots, (X_m, p_m(\cdot))),$$
there exists a unique catalog profile,
$$(C_1, \ldots, C_m),$$
such that for each firm $j \in I$
$$X_j = proj_X C_j$$
$$\text{and}$$
$$p_j(x) = \min\{p \in D : (x,p) \in C_j\} \text{ for all } x \in X_j,$$
and such that for all direct contracting mechanisms,
$$(i(\cdot), x(\cdot), p(\cdot)) \in \Sigma(C_1, \ldots, C_m),$$
and for agent types $t \in T$
$$i(t) \in I,\ x(t) \in X_{i(t)},\ \text{and}$$
$$p(t) = p_{i(t)}(x(t)).$$

By the competitive taxation principle, the pairing between nonlinear pricing schedules and catalogs is unique. Thus, in modeling problems of competitive nonlinear pricing, no loss of generality is introduced by focusing on catalogs rather than nonlinear pricing schedules.

2.2.4 The Firm's Expected Catalog Profit

For $t \in T$ and $(C_1, \ldots, C_m) \in \mathbf{P}$, let

$$\pi_j^*(t, C_1, \ldots, C_m) = \max\left\{\pi_j(i, x, p) : (i, x, p) \in \Phi(t, C_1, \ldots, C_m)\right\}, \tag{14}$$

where $\pi_j(\cdot, \cdot, \cdot)$ is the profit function specified in assumption (A-4). The quantity, $\pi_j^*(t, C_1, \ldots, C_m)$, is the maximum profit attainable by firm j given agent type t and catalog profile $(C_1, \ldots, C_m)$. Given assumption (A-4) and given the upper semicontinuity and measurability properties of the best response mapping (see the Proposition above), it follows from Proposition 4.3 in Page (1992) that the potential catalog profit function defined in expression (14) is upper semicontinuous on $\mathbf{P}$ and $B(T) \times B(\mathbf{P})$-measurable on $T \times \mathbf{P}$ (see Page (1992), p. 275).

Given catalog profile, $(C_1, \ldots, C_m) \in \mathbf{P}$, the j^{th} firm's expected maximal catalog profit is given by

$$\Pi_j^*(C_1, \ldots, C_m) = \int_T \pi_j^*(t, C_1, \ldots, C_m) d\mu(t). \tag{15}$$

It follows from Fatou's Lemma (see Aliprantis and Border (1999), p. 407), that each firm's expected maximal catalog profit function, $\Pi_j^*(\cdot, \ldots, \cdot)$ is $h_{\mathbf{P}}$-upper semicontinuous on $\mathbf{P}$.

If we take as the firm's payoff function the expected maximal catalog profit function, $\Pi_j^*(\cdot, \ldots, \cdot)$, then we are implicitly assuming that each firm behaves as if all ties will be broken in the firm's favor - that is, we are assuming that whenever the agent is indifferent between sales contracts of two or more firms each firm assumes that the agent will choose the firm's contract. While the expected maximal catalog profit function has desirable properties, including upper semicontinuity, here we wish to make a more realistic assumption concerning tie breaking. Thus, here we shall assume that if the agent is indifferent between the sales contracts of two or more firms, then the agent will choose one of the firms randomly with equal probability. Given the equal probability tie-breaking rule, we shall then assume that for each $(t, C_1, \ldots, C_m)$, each firm computes its payoff by weighting its maximal profit, $\pi_j^*(t, C_1, \ldots, C_m)$, by the probability that it receives its maximal profit. To begin, let

$$H(t, C_1, \ldots, C_m) := \left\{i \in I : \exists\ (x, p) \in X \times D \text{ such that } (i, x, p) \in \Phi(t, C_1, \ldots, C_m)\right\}. \tag{16}$$

Thus, if $i \in H(t, C_1, \ldots, C_m)$, then there is a sales contract, $(x, p) \in C_i$, offered by the i^{th} firm that is optimal for a type t agent. The set $H(t, C_1, \ldots, C_m)$ is the set of firms over which a type t agent is indifferent given catalog profile $(C_1, \ldots, C_m)$. Now let

$$|H(t, C_1, \ldots, C_m)| = \text{ the number of firms contained in } H(t, C_1, \ldots, C_m).$$

For each $(t, C_1, \ldots, C_m)$ such that $\pi_j^*(t, C_1, \ldots, C_m) \neq 0$, the firm's maximal profit, $\pi_j^*(t, C_1, \ldots, C_m)$, weighted by the probability that it receives its maximal profit is given by

$$\frac{\pi_j^*(t, C_1, \ldots, C_m)}{|H(t, C_1, \ldots, C_m)|}. \tag{17}$$

With these details in hand, we can write the j^{th} firm's expected profit as

$$\Pi_j\left(C_1, \ldots, C_m\right) = \int_T \frac{\pi_j^*(t, C_1, \ldots, C_m)}{|H(t, C_1, \ldots, C_m)|} d\mu(t). \tag{18}$$

Note that it follows from the definition of the maximal profit function $\pi_j^*(\cdot, \cdot, \ldots, \cdot)$ given in expressions (4) and (14) and the definition of the set $H(\cdot, \cdot, \ldots, \cdot)$ given in expression (16) that for any agent type t and catalog profile $(C_1, \ldots, C_m)$,

$$\left.\begin{array}{c} |H(t, C_1, \ldots, C_m)| > 1 \text{ implies that } \pi_j^*(t, C_1, \ldots, C_m) \geq 0 \text{ for all } j, \\ \text{or conversely that} \\ \pi_j^*(t, C_1, \ldots, C_m) < 0 \text{ for some } j \text{ implies that } |H(t, C_1, \ldots, C_m)| = 1. \end{array}\right\} \tag{19}$$

We shall use this fact later on in our proofs.[5]

2.2.5 Catalog Games and Nash Equilibrium

Given the catalog model specified in assumptions (A-1)-(A-4), the corresponding catalog game is given by $(P_f(K_j), \Pi_j)_{j=1}^m$ where the set of catalogs, $P_f(K_j)$, is the j^{th} firm's contracting strategy set and Π_j is the j^{th} firm's expected catalog profit function, given in expression (18).

Definition (Nash Equilibrium for Catalog Games)

A catalog profile

$$(C_1^*, \ldots, C_m^*) \in P_f(K_1) \times \cdots \times P_f(K_m)$$

is a Nash equilibrium for the catalog game $(P_f(K_j), \Pi_j)_{j=1}^m$ *if for all* $j = 1, 2, \ldots, m$

$$\Pi_j(C_j^*, C_{-j}^*) \geq \Pi_j(C_j, C_{-j}^*) \textit{for all } C_j \in P_f(K_j).$$

2.2.6 Mixed Catalog Games and Nash Equilibrium

Because the space of catalog profiles, $P_f(K_1) \times \cdots \times P_f(K_m)$, is not a vector space, the usual method of proving existence via a fixed point argument is not available for catalog games. Thus, in order to address the existence question, we must introduce mixed strategies (or probabilistic strategies) over catalogs and consider the mixed extension of the catalog game. For each firm $j = 1, 2, \ldots, m$, let $\Delta(P_f(K_j))$ denote the set of all probability measures defined on the feasible set of catalogs, $P_f(K_j)$. The strategy set $\Delta(P_f(K_j))$ is the j^{th} firm's mixed (or probabilistic) catalog strategy set. A mixed catalog strategy for firm j is a probability measure $\lambda_j \in \Delta(P_f(K_j))$

[5] If $j \in H(t, C_1, \ldots, C_m)$ and $|H(t, C_1, \ldots, C_m)| = 2$, so that the agent is indifferent between firm j and some other firm j', and if firm j's profit is negative if chosen by the agent, then it is optimal for firm j to send the agent to the other firm, j'. Given the definition of firm profit in expression (4), this would imply that

$$\pi_j^*(t, C_1, \ldots, C_m) = 0.$$

Thus, $\pi_j^*(t, C_1, \ldots, C_m) < 0$ for some j implies that $|H(t, C_1, \ldots, C_m)| = 1$.

having the following interpretation: if firm j chooses strategy $\lambda_j \in \Delta(P_f(K_j))$, then given any Borel measurable subset of catalogs, $\mathbf{E} \in B(P_f(K_j))$, the probability that a catalog contained in $\mathbf{E}$ will be selected under strategy λ_j is $\lambda_j(\mathbf{E})$. Since the feasible set of catalogs, $P_f(K_j)$, equipped the Hausdorff metric, is a compact metric space, the mixed catalog strategy set $\Delta(P_f(K_j))$ is convex, compact, and metrizable for the topology of weak convergence of measures (see Aliprantis and Border (1999), Chapter 14).

If firms choose mixed strategy profile

$$(\lambda_1, \ldots, \lambda_m) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m)),$$

then the j^{th} firm's expected payoff is given by

$$F_j(\lambda_1, \ldots, \lambda_m) := \int_{P_f(K_1)\times\cdots\times P_f(K_m)} \Pi_j(C_1, \ldots, C_m)\, \lambda_1(dC_1) \cdot \cdots \cdot \lambda_m(dC_m). \quad (20)$$

Following the terminology of Reny (1999), the game $(\Delta(P_f(K_j)), F_j)_{j=1}^m$ represents the mixed extension of the underlying catalog game $(P_f(K_j), \Pi_j)_{j=1}^m$. We shall refer to the game $(\Delta(P_f(K_j)), F_j)_{j=1}^m$ as the mixed catalog game. Note that for each firm j the expected payoff function,

$$\lambda_j \to F_j(\lambda_j, \lambda_{-j}),$$

is linear on the strategy space $\Delta(P_f(K_j))$ for each $\lambda_{-j} \in \Delta_{-j}(P_f(K_{-j}))$ (see Aliprantis and Border (1999), Theorem 14.5, p. 479).[6]

Definition (Nash Equilibrium for Mixed Catalog Games)

A strategy profile

$$(\lambda_1^*, \ldots, \lambda_m^*) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m))$$

is a Nash equilibrium for the mixed catalog game

$$(\Delta(P_f(K_j)), F_j)_{j=1}^m$$

if for all $j = 1, 2, \ldots, m$

$$F_j\left(\lambda_j^*, \lambda_{-j}^*\right) \geq F_j\left(\lambda_j, \lambda_{-j}^*\right) \textit{ for all } \lambda_j \in \Delta(P_f(K_j)).$$

3 The Existence of Nash Equilibrium for Mixed Catalog Games

3.1 Main Result

Theorem 2 *(Existence of Nash Equilibrium for Mixed Catalog Games):*

Under assumptions (A-1)-(A-4), the mixed catalog game, $(\Delta(P_f(K_j)), F_j)_{j=1}^m$*, has a Nash equilibrium.*

[6] Here, $\Delta_{-j}(P_f(K_{-j})) = \prod_{i \neq j} \Delta(P_f(K_i))$.

By the competitive taxation principle, each nonlinear pricing schedule, $(X_j, p_j(\cdot))$, is uniquely indexed by a product-price catalog $C_j \in P_f(K_j)$. Thus, the mixed catalog game, $(\Delta(P_f(K_j)), F_j)_{j=1}^m$, played over product-price catalogs can be viewed as the mixed extension of the nonlinear pricing game played over the index set, $P_f(K_1) \times \cdots \times P_f(K_m)$. If under the Nash equilibrium profile of mixed strategies,

$$(\lambda_1^*, \ldots, \lambda_m^*),$$

product-price catalog profile $(C_1, \ldots, C_m)$ is chosen, then this is equivalent to choosing the profile of nonlinear pricing schedules,

$$((X_1, p_1(\cdot)), \ldots, (X_m, p_m(\cdot))),$$

uniquely indexed by $(C_1, \ldots, C_m)$.

4 Proofs

By Theorem 3.1 and Corollary 5.2 in Reny (1999), it suffices to show that under assumptions (A-1)-(A-4) the mixed catalog game, $(\Delta(P_f(K_j)), F_j)_{j=1}^m$, is better-reply secure. We shall proceed by first proving a series of three propositions. These propositions will then allow us to easily establish the better-reply security of mixed catalog games.

To begin, let

$$graphF(\cdot) := \{(\lambda, F) : F = F(\lambda)\},$$

where $\lambda = (\lambda_1, \ldots, \lambda_m) \in \prod_j \Delta(P_f(K_j))$ and $F = (F_1, \ldots, F_m) \in R^m$, and where

$$F(\lambda) := (F_1(\lambda), \ldots, F_m(\lambda)),$$

and for $j = 1, 2, \ldots, m$,

$$\begin{aligned} F_j(\lambda) &:= F_j(\lambda_1, \ldots, \lambda_m) \\ &:= \int_{P_f(K_1)\times\cdots\times P_f(K_m)} \Pi_j(C_1, \ldots, C_m)\, \lambda_1(dC_1) \cdot \cdots \cdot \lambda_m(dC_m). \end{aligned}$$

Letting cl denote closure, we have

$$(\lambda, F) \in cl\,\{graphF(\cdot)\},$$

if and only if there exists a sequence of mixed strategy profiles, $\{\lambda^n\}_{n=1}^\infty$ contained in $\prod_j \Delta(P_f(K_j))$ such that λ^n converges to λ in metrizable topology of weak convergence of measures, and $F(\lambda^n)$ converges to F in R^m.

Better-reply Security (Mixed Strategies): *We say that the mixed catalog game, $(\Delta(P_f(K_j)), F_j)_{j=1}^m$, is better-reply secure if for every mixed strategy profile*

$$\lambda' = (\lambda_1', \ldots, \lambda_m') \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m)),$$

and every

$$F' = \left(F'_1, \ldots, F'_m\right) \in R^m,$$

such that

$$(\lambda', F') \in cl\left\{graphF(\cdot)\right\},$$

and λ' *is not a Nash equilibrium, then there exists some firm* $j \in \{1, \ldots, m\}$, *a* $\delta > 0$, *and mixed strategies* $\lambda_j^* \in \Delta(P_f(K_j))$ *such that*

$$F_j\left(\lambda_j^*, \lambda_{-j}\right) > F'_j.$$

for all $(m-1)$*-tuples of mixed strategies* $\lambda_{-j} \in B_\delta(\lambda'_{-j})$.

Here, $B_\delta(\lambda'_{-j})$ is an open ball of radius δ centered at λ'_{-j} in the metric space of probability measures, $\Delta_{-j}(P_f(K_{-j}))$.

Thus, the mixed catalog game, $(\Delta(P_f(K_j)), F_j)_{j=1}^m$, is better-reply secure if starting at any pair

$$(\lambda', F') \in cl\left\{graphF(\cdot)\right\},$$

where $\lambda' = \left(\lambda'_1, \ldots, \lambda'_m\right)$ is not a Nash equilibrium, there is a firm j which has a mixed strategy, λ_j^*, it can move to in order to secure an expected payoff greater than F'_j if other firms begin to deviate from their mixed strategies, λ'_{-j}, to other mixed strategies in a neighborhood of λ'_{-j}.

4.1 Proposition 1: Catalog Games Are Uniformly Payoff Secure

We begin by showing that under assumptions (A-1)-(A-4), the catalog game,

$$(P_f(K_j), \Pi_j)_{j=1}^m,$$

is uniformly payoff secure in pure catalog strategies. Uniform payoff security, a notion introduced in Page and Monteiro (2003), is *critical* to showing that catalog games are better-reply secure. The formal definition of uniform payoff security is as follows:

Uniform Payoff Security (Pure Strategies): *We say that the catalog game,* $(P_f(K_j), \Pi_j)_{j=1}^m$, *is uniformly payoff secure if for every firm* $j \in \{1, \ldots, m\}$, *every catalog* $C_j^* \in P_f(K_j)$, *and every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *and a catalog* $\widehat{C}_j \in P_f(K_j)$ *such that*

$$\Pi_j(\widehat{C}_j, C'_{-j}) \geq \Pi_j(C_j^*, C_{-j}) - \varepsilon$$

for all $(m-1)$*-tuples of catalogs* C'_{-j} *and* C_{-j} *such that* $h_{\mathbf{P}_{-j}}(C'_{-j}, C_{-j}) < \delta$.

Here, $h_{\mathbf{P}_{-j}}(C'_{-j}, C_{-j})$ is given by

$$h_{\mathbf{P}_{-j}}(C'_{-j}, C_{-j}) := \max\{h(C'_i, C_i) : i \neq j\},$$

and defines a metric on the space of partial catalog profiles,

$$P_f(K_{-j}) := P_f(K_1) \times \cdots \times P_f(K_{j-1}) \times P_f(K_{j+1}) \times \cdots \times P_f(K_m).$$

Thus, the catalog game, $(P_f(K_j), \Pi_j)_{j=1}^m$, is payoff secure if for each firm j and each starting catalog strategy C_j^* there is a defensive catalog strategy, $\widehat{C}_j$, that firm j can move to in order to secure an expected payoff of at least $\Pi_j(C_j^*, C_{-j}) - \varepsilon$ given any starting strategy profile, C_{-j}, of other firms and any deviation by other firms to strategies, C'_{-j}, in a neighborhood of C_{-j}. Note that under uniform payoff security, the neighborhood size δ of other firms' deviations is invariant with respect to other firms' starting strategies, C_{-j}.

Proposition 3 *(Catalog Games Are Uniformly Payoff Secure):*
Under assumptions (A-1)-(A-4), the catalog game, $(P_f(K_j), \Pi_j)_{j=1}^m$, is uniformly payoff secure.

Proof. : Let

$$\Sigma(C_1, \ldots, C_m)$$

denote the set of all measurable selectors from the best response mapping

$$t \to \Phi(t, C_1, \ldots, C_m).$$

By the Kuratowski-Ryll-Nardzewski Selection Theorem, $\Sigma(C_1, \ldots, C_m)$ is nonempty for any catalog profile $(C_1, \ldots, C_m)$ (see Aliprantis and Border (1999), p. 567).

Let $(C_1^*, C_2, \ldots, C_m)$ be given and let

$$(i^*(\cdot), x^*(\cdot), p^*(\cdot)) \in \Sigma(C_1^*, C_2, \ldots, C_m)$$

be such that

$$\Pi_j^*(C_1^*, C_2, \ldots, C_m) = \int_T (p^*(t) - c_1(x^*(t)))\, I_1(i^*(t)) d\mu(t).$$

Thus, $(i^*(\cdot), x^*(\cdot), p^*(\cdot))$ is an optimal measurable selector for firm 1. Let $\varepsilon > 0$ be given and fix $\delta > 0$ so that for each $j \in \{1, 2, \ldots, m\}$, $d((x_j, p_j), (x'_j, p'_j)) < \delta$, implies that $\left|(u(t, j, x_j) - p_j) - \left(u(t, j, x'_j) - p'_j\right)\right| < \varepsilon$ for all $t \in T$. Given $\varepsilon > 0$, we can now define a defensive catalog, $\widehat{C}_1$, corresponding to starting catalog C_1^*. In particular, let

$$\widehat{C}_1 := \{(x, p - \varepsilon) : (x, p) \in C_1^* \text{ and } (p - c_1(x)) \geq \varepsilon\}.$$

Also, let

$$T_1^* := \{t \in T : i^*(t) = 1 \text{ and } p^*(t) - c_1(x^*(t)) \geq \varepsilon\}.$$

T_1^* is the set of agent types choosing from firm 1's catalog C_1^* and generating profit of $p^*(t) - c_1(x^*(t)) \geq \varepsilon$. Now consider the catalog deviation $(C'_2, \ldots, C'_m)$ by other firms such that for each $j = 2, 3, \ldots, m$, $h(C'_j, C_j) < \delta$, where $(C_2, \ldots, C_m)$ is the starting catalog profile of other firms. By the properties of the Hausdorff metric

h, for each $j = 2, 3, \ldots, m$ and $(x'_j, p'_j) \in C'_j$ there exists $(x_j, p_j) \in C_j$ such that $d((x'_j, p'_j), (x_j, p_j)) < \delta$. Thus, by equicontinuity (A-3)(iii), we have for each $j = 2, 3, \ldots, m$ and all $t \in T$,

$$u(t, j, x'_j) - p'_j < u(t, j, x_j) - p_j + \varepsilon = u(t, j, x_j) - (p_j - \varepsilon),$$

and for each $j = 2, 3, \ldots, m$ and $t \in T_1^*$, we have

$$u(t, j, x'_j) - p'_j < u(t, j, x_j) - (p_j - \varepsilon) \leq u(t, i^*(t), x^*(t)) - (p^*(t) - \varepsilon),$$

where, by the definition of T_1^*, $(x^*(t), p^*(t) - \varepsilon) \in \widehat{C}_1$.

Now let

$$(\widehat{i}(\cdot), \widehat{x}(\cdot), \widehat{p}(\cdot)) \in \Sigma(\widehat{C}_1, C'_2, \ldots, C'_m)$$

be such that

$$\Pi_1^*(\widehat{C}_1, C'_2, \ldots, C'_m) = \int_T (\widehat{p}(t) - c_1(\widehat{x}(t)))\, I_1(\widehat{i}(t)) d\mu(t).$$

Since for each $t \in T_1^*$, $(x^*(t), p^*(t) - \varepsilon) \in \widehat{C}_1$, we have for each $j = 2, 3, \ldots, m$ and $t \in T_1^*$, $\widehat{i}(t) = 1$ and

$$\left.\begin{aligned} & u(t, j, x'_j) - p'_j \\ & < u(t, j, x_j) - (p_j - \varepsilon) \\ & \leq u(t, i^*(t), x^*(t)) - (p^*(t) - \varepsilon) \\ & \leq u(t, \widehat{i}(t), \widehat{x}(t)) - \widehat{p}(t). \end{aligned}\right\} \quad (*)$$

Inequality $(*)$ implies that for $t \in T_1^*$ and catalog profile $(\widehat{C}_1, C'_2, \ldots, C'_m)$,

$$H(t, \widehat{C}_1, C'_2, \ldots, C'_m) = \{1\}.$$

We have therefore,

$$\begin{aligned} & \Pi_1 (C_1^*, C_2, \ldots, C_m) - \varepsilon = \int_T \tfrac{\pi_1^*(t, C_1^*, C_2, \ldots, C_m)}{|H(t, C_1, \ldots, C_m)|} d\mu(t) - \varepsilon \\ & \leq \int_T \tfrac{\pi_1^*(t, C_1^*, C_2, \ldots, C_m) - \varepsilon}{|H(t, C_1, \ldots, C_m)|} d\mu(t) \\ & = \int_T \tfrac{(p^*(t) - c_1(x^*(t))) I_1(i^*(t)) - \varepsilon}{|H(t, C_1, \ldots, C_m)|} d\mu(t) \\ & \leq \int_{T_1^*} \tfrac{(p^*(t) - \varepsilon - c_1(x^*(t))) I_1(i^*(t))}{\left|H(t, \widehat{C}_1, C'_2, \ldots, C'_m)\right|} d\mu(t) \\ & \leq \int_{T_1^*} \tfrac{(\widehat{p}(t) - c_1(\widehat{x}(t))) I_1(\widehat{i}(t))}{\left|H(t, \widehat{C}_1, C'_2, \ldots, C'_m)\right|} d\mu(t) \\ & \leq \int_T \tfrac{(\widehat{p}(t) - c_1(\widehat{x}(t))) I_1(\widehat{i}(t))}{\left|H(t, \widehat{C}_1, C'_2, \ldots, C'_m)\right|} d\mu(t) = \Pi_1(\widehat{C}_1, C'_2, \ldots, C'_m). \end{aligned}$$

■

Theorem 5 in Page and Monteiro (2003) established that, under assumptions (A-1)-(A-4), the catalog game,

$$\left(P_f(K_j), \Pi_j^*\right)_{j=1}^m,$$

where expected catalog profit, Π_j^*, is given by expression (15) is uniformly payoff secure. Here we have established uniform payoff security for the more difficult case where expected catalog profit, Π_j, is given by expression (18). It follows by observation that any game that is uniformly payoff secure is automatically payoff secure.

By Proposition 3.2 in Reny (1999), if in addition to being payoff secure the catalog game is reciprocally upper semicontinuous (u.s.c.), then it is better-reply secure. A catalog game, $(P_f(K_j), \Pi_j)_{j=1}^m$, is reciprocally upper semicontinuous if whenever $(C', \Pi') \in cl\,\{graph\Pi(\cdot)\}$ and

$$\Pi_j(C') \leq \Pi_j' \text{ for all } j = 1, 2, \ldots, m,$$

then

$$\Pi_j(C') = \Pi_j' \text{ for all } j = 1, 2, \ldots, m.$$

Here

$$graph\Pi(\cdot) := \{(C, \Pi) : \Pi = \Pi(C)\},$$

where $C = (C_1, \ldots, C_m) \in P_f(K_1) \times \cdots \times P_f(K_m)$. and $\Pi = (\Pi_1, \ldots, \Pi_m) \in R^m$, and where

$$\Pi(C) := (\Pi_1(C), \ldots, \Pi_m(C)).$$

Reciprocal u.s.c. was introduced by Simon (1987) under the name complementary discontinuities. It requires that some firm's expected catalog profit jump up whenever some other firm's catalog profit jumps down. For the catalog game where expected catalog profit given by

$$\Pi_j^*(C_1, \ldots, C_m) = \int_T \pi_j^*(t, C_1, \ldots, C_m)\, d\mu(t),$$

reciprocal u.s.c. is satisfied automatically because $\Pi_j^*(\cdot, \ldots, \cdot)$ is upper semicontinuous in catalog profiles. However, as the following example will illustrate, for catalog games where expected catalog profit is given by

$$\Pi_j(C_1, \ldots, C_m) = \int_T \frac{\pi_j^*(t, C_1, \ldots, C_m)}{|H(t, C_1, \ldots, C_m)|}\, d\mu(t),$$

reciprocal u.s.c. fails to hold in general.

Example 1 *(The Failure of Reciprocal u.s.c.): Suppose there are two firms and consider the sequence of catalogs given by*

$$\{(C_1^n, C_2^n)\}_{n=1}^\infty = \left\{\left(1, p - \frac{1}{n}\right), (1, p)\right\}_{n=1}^\infty$$

Also, suppose that firm 1 has profit $r\left(p-\frac{1}{n}\right)$ *when chosen by the agent while firm 2 has zero profit when chosen by the agent. Finally, suppose that the agent has utility function given by*

$$v(t,i,x,p)=1-p.$$

For all n*, the agent chooses from firm 1. Thus,*

$$\Pi_1\left(C_1^n,C_2^n\right)=r\left(p-\tfrac{1}{n}\right)\rightarrow rp$$
while
$$\Pi_2\left(C_1^n,C_2^n\right)=0 \text{ for all } n.$$

In the limit

$$\left(C_1^n,C_2^n\right)\rightarrow\left\{\left(\bar{C}_1,\bar{C}_2\right)\right\}=\left\{(1,p),(1,p)\right\},$$

and the agent is indifferent between firms 1 and 2. Thus, in the limit $\Pi_1\left(\bar{C}_1,\bar{C}_2\right)=\frac{rp}{2}<rp$ *and* $\Pi_2\left(\bar{C}_1,\bar{C}_2\right)=0$ *- and thus, reciprocal u.s.c. fails.*

In light of this example, we cannot use Reny's Proposition 3.2 to establish better-reply security for catalog games $(P_f(K_j),\Pi_j)_{j=1}^m$ with Π_j given by expression (18). Nor can we use Reny's Proposition 3.2 to establish better-reply security for mixed catalog games $(\Delta(P_f(K_j)),F_j)_{j=1}^m$. However, using uniform payoff security and methods used in the proof of our Proposition 1, as well as our next proposition establishing the payoff security of mixed catalog games, we will be able to establish better reply security for catalog games and their mixed extensions.

4.2 Proposition 2: Mixed Catalog Games Are Payoff Secure

We begin by formally stating the notion of payoff security introduced by Reny (1999). This condition is crucial to proving that mixed catalog games are better-reply secure - even without reciprocal u.s.c. and Reny's Proposition 3.2.

Payoff Security (Mixed Strategies): *We say that the mixed catalog game,* $(\Delta(P_f(K_j)),F_j)_{j=1}^m$, *is payoff secure if for every mixed strategy profile*

$$\left(\lambda_1',\ldots,\lambda_m'\right)\in\Delta(P_f(K_1))\times\cdots\times\Delta(P_f(K_m)),$$

and every $\varepsilon>0$ *there exists a* $\delta>0$ *and mixed strategies* $\lambda_j^*\in\Delta(P_f(K_j))$, $j=1,\ldots,m$, *such that*

$$F_j\left(\lambda_j^*,\lambda_{-j}\right)\geq F_j\left(\lambda_j',\lambda_{-j}'\right)-\varepsilon.$$

for all $(m-1)$*-tuples of mixed strategies* $\lambda_{-j}\in B_\delta(\lambda_{-j}')$.

Here, $B_\delta(\lambda_{-j}')$ is an open ball of radius δ centered at λ_{-j}' in the metric space of probability measures,

$$\Delta_{-j}(P_f(K_{-j})):=\Delta(P_f(K_1))\times\cdots\times\Delta(P_f(K_{j-1}))\times\Delta(P_f(K_{j+1}))\times\cdots\times\Delta(P_f(K_m)).$$

Proposition 4 *(Mixed Catalog Games Are Payoff Secure):*

Under assumptions (A-1)-(A-4), the mixed catalog game, $(\Delta(P_f(K_j)), F_j)_{j=1}^m$*, is payoff secure.*

Proof. Let

$$(\lambda_1^*, \ldots, \lambda_m^*) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m)),$$

be any mixed strategy profile and consider firm 1. Firm 1's expected payoff is given by

$$F_1(\lambda_1^*, \ldots, \lambda_m^*) := \int_{P_f(K_1)\times\cdots\times P_f(K_m)} \Pi_1(C_1, \ldots, C_m)\, \lambda_1^*(dC_1)\cdot\cdots\cdot\lambda_m^*(dC_m).$$

Let $\varepsilon^* > 0$ be given, We will show that there exists a mixed strategy λ_1' and a $\delta > 0$ such that if

$$\mu_{-1} = \mu_2 \times \ldots \times \mu_m \in B_\delta(\lambda_{-1}^*)$$

then

$$F_1\left(\lambda_1', \mu_2, \ldots, \mu_m\right) \geq F_1(\lambda_1^*, \ldots, \lambda_m^*) - \varepsilon^*.$$

As before, $B_\delta(\lambda_{-1}^*)$ is an open ball of radius δ centered at λ_{-1}^* in the metric space of probability measures,

$$\Delta_{-1}(P_f(K_{-1})) := \Delta(P_f(K_2)) \times \cdots \times \Delta(P_f(K_m)).$$

First let $\varepsilon > 0$ be given and choose catalog $C_1^* \in P_f(K_1)$ so that

$$F_1(C_1^*, \lambda_2^*, \ldots, \lambda_m^*) \geq \sup_{C_1\in P_f(K_1)} F_1(C_1, \lambda_2^*, \ldots, \lambda_m^*) - \varepsilon.$$

Since

$$\sup_{C_1\in P_f(K_1)} F_1(C_1, \lambda_2^*, \ldots, \lambda_m^*) - \varepsilon \geq F_1(\lambda_1^*, \ldots, \lambda_m^*) - \varepsilon,$$

we have

$$F_1(C_1^*, \lambda_2^*, \ldots, \lambda_m^*) \geq F_1(\lambda_1^*, \ldots, \lambda_m^*) - \varepsilon. \tag{21}$$

By the UPS (uniform payoff security) property of the catalog game, given C_1^* and $\varepsilon > 0$, there exists $\delta > 0$ and $\widehat{C}_1$ such for all C_{-1}' and C_{-1} with $C_{-1}' \in B_\delta(C_{-1})$

$$\Pi_1(\widehat{C}_1, C_{-1}') \geq \Pi_1(C_1^*, C_{-1}) - \varepsilon.$$

Now observe that the collection of open balls,

$$\left\{B_{\frac{\delta}{2}}(C_{-1})\right\}_{C_{-1}\in P_f(K_{-1})},$$

where each open ball, $B_{\frac{\delta}{2}}(C_{-1})$, has radius $\frac{\delta}{2} > 0$ and is centered at C_{-1}, forms an open cover of the compact metric space $P_f(K_{-1})$. Thus, there is a finite subcover of $P_f(K_{-1})$, denoted by

$$\left\{B_{\frac{\delta}{2}}(C_{-1}^h)\right\}_{h\in H}$$

where $H = \{1, 2, \ldots, h^*\}$.

Because λ^*_{-1} is a finite measure on $P_f(K_{-1})$, for each h we can choose a radius r_h such that

$$\frac{\delta}{2} < r_h < \delta,$$

such that

$$\lambda^*_{-1}(\partial B_{r_h}(C^h_{-1})) = 0 \text{ for all } h.$$

Here, $\partial B_{r_h}(C^h_{-1})$ denotes the boundary of the open ball $B_{r_h}(C^h_{-1})$.

By UPS we have for each h

$$\Pi_1\left(\widehat{C}_1, C'_{-1}\right) \geq \Pi_1(C^*_1, C_{-1}) - \varepsilon \text{ for all } C'_{-1} \text{ and } C_{-1} \text{ contained in } B_{r_h}(C^h_{-1}). \quad (22)$$

For each $h \in H$, choose C^{*h}_{-1} so that

$$\Pi_1\left(C^*_1, C^{*h}_{-1}\right) \geq \sup_{C_{-1} \in B_{r_h}(C^h_{-1})} \Pi_1\left(C^*_1, C_{-1}\right) - \varepsilon. \quad (23)$$

Therefore, by (22) and (23) we have

$$\Pi_1\left(\widehat{C}_1, C'_{-1}\right) \geq \Pi_1(C^*_1, C^{*h}_{-1}) - \varepsilon \text{ for all } C'_{-1} \in B_{r_h}(C^h_{-1}). \quad (24)$$

Define

$$D^1 = \overline{B}_{r_1}(C^1_{-1}),\ D^2 = \overline{B}_{r_2}(C^2_{-1}) \setminus \overline{B}_{r_1}(C^1_{-1}), \ldots, D^{h^*} = \overline{B}_{r_{h^*}}(C^{h^*}_{-1}) \setminus \bigcup_{h=1}^{h^*} \overline{B}_{r_h}(C^h_{-1}).$$

Thus,

$\int_{P_f(K_{-1})} \Pi_1\left(\widehat{C}_1, C'_{-1}\right) d\mu_{-1}\left(C'_{-1}\right) = \sum_{h=1}^{h^*} \int_{D^h} \Pi_1\left(\widehat{C}_1, C'_{-1}\right) d\mu_{-1}\left(C'_{-1}\right)$

$\geq \sum_{h=1}^{h^*} \int_{D^h} \left(\Pi_1(C_1^*, C_{-1}^{*h}) - \varepsilon\right) d\mu_{-1}\left(C_{-1}\right)$

$= \sum_{h=1}^{h^*} \Pi_1(C_1^*, C_{-1}^{*h})\mu_{-1}\left(D^h\right) - \varepsilon$

$= \sum_{h=1}^{h^*} \Pi_1(C_1^*, C_{-1}^{*h})\mu_{-1}\left(D^h\right)$
$+ \left(F_1\left(\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*\right) - 2\varepsilon - \left(F_1\left(\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*\right) - \varepsilon\right)\right)$

$\geq \sum_{h=1}^{h^*} \Pi_1(C_1^*, C_{-1}^{*h})\mu_{-1}\left(D^h\right)$
$+ \left(F_1\left(\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*\right) - 2\varepsilon - F_1\left(C_1^*, \lambda_2^*, \ldots, \lambda_m^*\right)\right)$

$= \sum_{h=1}^{h^*} \Pi_1(C_1^*, C_{-1}^{*h})\mu_{-1}\left(D^h\right)$
$+ \left(F_1\left(\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*\right) - 2\varepsilon - \sum_{h=1}^{h^*} \int_{D^h} \Pi_1\left(C_1^*, C_{-1}\right) d\lambda_{-1}^*\left(C_{-1}\right)\right)$

$= \sum_{h=1}^{h^*} \Pi_1(C_1^*, C_{-1}^{*h})\mu_{-1}\left(D^h\right)$
$+ \left(F_1\left(\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*\right) - 3\varepsilon - \sum_{h=1}^{h^*} \int_{D^h} \left(\Pi_1\left(C_1^*, C_{-1}\right) - \varepsilon\right) d\lambda_{-1}^*\left(C_{-1}\right)\right)$

$\geq \sum_{h=1}^{h^*} \Pi_1(C_1^*, C_{-1}^{*h})\mu_{-1}\left(D^h\right)$
$+ \left(F_1\left(\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*\right) - 3\varepsilon - \sum_{h=1}^{h^*} \int_{D^h} \Pi_1\left(C_1^*, C_{-1}^{*h}\right) d\lambda_{-1}^*\left(C_{-1}\right)\right)$

$= \sum_{h=1}^{h^*} \Pi_1(C_1^*, C_{-1}^{*h})\mu_{-1}\left(D^h\right)$
$+ \left(F_1\left(\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*\right) - 3\varepsilon - \sum_{h=1}^{h^*} \Pi_1(C_1^*, C_{-1}^{*h})\lambda_{-1}^*\left(D^h\right)\right)$

$\geq F_1\left(\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*\right) - 3\varepsilon - h^* \left|\Pi_1\right|_\infty \max_h \left|\mu_{-1}\left(D^h\right) - \lambda_{-1}^*\left(D^h\right)\right|.$

Since the boundary of D^h has measure zero for all h, for any $\varepsilon > 0$ we can choose a $\gamma_\varepsilon > 0$ such that if $\mu_{-1} \in B_{\gamma_\varepsilon}(\lambda_{-1}^*)$, then

$$h^* \left|\Pi_1\right|_\infty \max_h \left|\mu_{-1}\left(D^h\right) - \lambda_{-1}^*\left(D^h\right)\right| < \varepsilon,$$

(see Ash (1972), Theorem 4.5.1(c), p. 196). Here, $\left|\Pi_1\right|_\infty = \sup_C \Pi_1(C)$. Thus, we have for all $\mu_{-1} \in B_{\gamma_\varepsilon}(\lambda_{-1}^*)$,

$$F_1(\widehat{C}_1, \mu_{-1}) = \int_{P_f(K_{-1})} \Pi_1\left(\widehat{C}_1, C'_{-1}\right) d\mu_{-1}\left(C'_{-1}\right) \geq F_1\left(\lambda_1^*, \ldots, \lambda_m^*\right) - 4\varepsilon.$$

Let $\lambda'_1 \in \Delta(P_f(K_1))$ be such that

$$\lambda'_1(\{\widehat{C}_1\}) = 1.$$

Then we have

$$F_1(\lambda_1', \mu_2, \ldots, \mu_m) \geq F_1\left(\lambda_1^*, \ldots, \lambda_m^*\right) - 4\varepsilon \text{ for all } \mu_{-1} \in B_{\gamma_\varepsilon}(\lambda_{-1}^*).$$

■

Theorem 6 in Page and Monteiro (2003) established that, under assumptions (A-1)-(A-4), the mixed catalog game,

$$\left(\Delta(P_f(K_j)), F_j^*\right)_{j=1}^m,$$

with underlying expected catalog profit, Π_j^*, given by expression (15) is payoff secure. The proof above is modified version of the proof of Theorem 6 in Page and Monteiro (2003) - where the modifications made are those needed to accommodate the change from expected catalog profit being given by, Π_j^*, expression (15) to expected catalog profit being given by, Π_j, in expression (18).

4.3 Proposition 3: Catalog Games Are Better-Reply Secure

Before proving our main result - that mixed catalog games are better-reply secure - we will show that catalog games are better-reply secure. We shall then use this result and some parts of the method of proof to establish our main result.

Proposition 5 *(Catalog Games Are Better-Reply Secure):*
Under assumptions (A-1)-(A-4), the catalog game, $(P_f(K_j), \Pi_j)_{j=1}^m$, *is better-reply secure.*

Proof. Let $\{C^n\}_n$ be a sequence of catalog profiles $h_{\mathbf{P}}$-converging to catalog profile C that is not a Nash equilibrium and suppose that

$$\Pi_j(C^n) \to \Pi_j \text{ for all } j.$$

For all j we have,

$$\Pi_j = \lim_n \Pi_j(C^n)$$

$$= \lim_n \int_T \frac{\pi_j^*(t, C^n)}{|H(t, C^n)|} d\mu(t)$$

$$\leq \int_T \limsup_n \frac{\pi_j^*(t, C^n)}{|H(t, C^n)|} d\mu(t)$$

$$\leq \int_T \limsup_n \pi_j^*(t, C^n) d\mu(t)$$

$$\leq \int_T \pi_j^*(t, C) d\mu(t).$$

The last inequality holds by the upper semicontinuity of $\pi_j^*(t, \cdot)$, the second to last inequality holds due to the observations given in expression (19).

Let

$$T_j^+ = \left\{t \in T : \pi_j^*(t, C) \geq 0\right\}.$$

Case 1: Suppose that for some $j = 1, 2, \ldots, m$, $\mu(T \backslash T_j^+) > 0$. We have then

$$\Pi_j \leq \int_T \pi_j^*(t, C) d\mu(t) < \int_{T_j^+} \pi_j^*(t, C) d\mu(t).$$

Thus, for some $\varepsilon > 0$, we have

$$\Pi_j < \int_{T_j^+} \pi_j^*(t, C) d\mu(t) - \varepsilon.$$

Let

$$(i(\cdot), x(\cdot), p(\cdot)) \in \Sigma(C) = \Sigma(C_1, C_2, \ldots, C_m)$$

be such that

$$\Pi_j^*(C_1, C_2, \ldots, C_m) = \int_T (p(t) - c_j(x(t))) \, I_j(i(t)) d\mu(t).$$

We have

$$\Pi_j^*(C_1, C_2, \ldots, C_m) - \varepsilon = \int_T (p(t) - c_j(x(t))) \, I_j(i(t)) d\mu(t) - \varepsilon$$

$$\leq \int_{T_j^+} (p(t) - c_j(x(t))) \, I_j(i(t)) d\mu(t) - \varepsilon$$

$$\leq \int_{T_j^+} (p(t) - \varepsilon - c_j(x(t))) \, I_j(i(t)) d\mu(t).$$

Let

$$C_j^\varepsilon := \{(x, p - \varepsilon) : (x, p) \in C_j \text{ and } (p - c_j(x)) \geq \varepsilon\},$$

and

$$T_j^\varepsilon := \{t \in T : i(t) = j \text{ and } p(t) - c_j(x(t)) \geq \varepsilon \}.$$

We have $T_j^\varepsilon \subseteq T_j^+$ and

$$\int_{T_j^+} (p(t) - \varepsilon - c_j(x(t))) \, I_j(i(t)) d\mu(t) \leq \int_{T_j^\varepsilon} (p(t) - \varepsilon - c_j(x(t))) \, I_j(i(t)) d\mu(t).$$

Given $\varepsilon > 0$, fix $\delta > 0$ so that for each $j \in \{1, 2, \ldots, m\}$, $d((x_j, p_j), (x'_j, p'_j)) < \delta$, implies that $\left|(u(t, j, x_j) - p_j) - \left(u(t, j, x'_j) - p'_j\right)\right| < \varepsilon$ for all $t \in T$. Now consider catalog deviations C'_{-j} by firms other than j such that for each $j' \neq j$, $h(C'_{j'}, C_{j'}) < \delta$. By the properties of the Hausdorff metric h, for each $j' \neq j$, and $(x'_{j'}, p'_{j'}) \in C'_{j'}$ there exists $(x_{j'}, p_{j'}) \in C_{j'}$ such that $d((x'_{j'}, p'_{j'}), (x_{j'}, p_{j'})) < \delta$. Thus, by equicontinuity (A-3)(iii), we have for $j' \neq j$ and all $t \in T$,

$$u(t, j', x'_{j'}) - p'_{j'} < u(t, j', x_{j'}) - p_{j'} + \varepsilon = u(t, j', x_{j'}) - (p_{j'} - \varepsilon),$$

and for each $j' \neq j$ and $t \in T_j^\varepsilon$, we have

$$u(t, j', x'_{j'}) - p'_{j'} < u(t, j', x_{j'}) - (p_{j'} - \varepsilon) \leq u(t, i(t), x(t)) - (p(t) - \varepsilon),$$

where, by the definition of T_j^{ε}, $(x(t), p(t) - \varepsilon) \in C_j^{\varepsilon}$.

Now let

$$(i^{\varepsilon}(\cdot), x^{\varepsilon}(\cdot), p^{\varepsilon}(\cdot)) \in \Sigma(C_j^{\varepsilon}, C'_{-j})$$

be such that

$$\Pi_j^*(C_j^{\varepsilon}, C'_{-j}) = \int_T (p^{\varepsilon}(t) - c_j(x^{\varepsilon}(t)))\, I_j(i^{\varepsilon}(t)) d\mu(t).$$

Since for each $t \in T_j^{\varepsilon}$, $(x(t), p(t) - \varepsilon) \in C_j^{\varepsilon}$, we have for each $j' \neq j$ and $t \in T_j^{\varepsilon}$, $i^{\varepsilon}(t) = j$ and

$$\left.\begin{aligned} & u(t, j', x'_{j'}) - p'_{j'} \\ & < u(t, j', x_{j'}) - (p_{j'} - \varepsilon) \\ & \leq u(t, i(t), x(t)) - (p(t) - \varepsilon) \\ & \leq u(t, i^{\varepsilon}(t), x^{\varepsilon}(t)) - p^{\varepsilon}(t). \end{aligned}\right\} \quad (**)$$

Inequalities $(**)$ imply that for $t \in T_j^{\varepsilon}$ and catalog profile $(C_j^{\varepsilon}, C'_{-j})$,

$$H(t, (C_j^{\varepsilon}, C'_{-j})) = \{j\}.$$

We have therefore,

$$\begin{aligned} & \textstyle\int_{T_j^+} (p(t) - \varepsilon - c_j(x(t)))\, I_j(i(t)) d\mu(t) \\ & \leq \textstyle\int_{T_j^{\varepsilon}} (p(t) - \varepsilon - c_j(x(t)))\, I_j(i(t)) d\mu(t) \\ & \leq \textstyle\int_{T_j^{\varepsilon}} \frac{\pi_1^*(t, C_j^{\varepsilon}, C'_{-j})}{\left|H(t, C_j^{\varepsilon}, C'_{-j})\right|} d\mu(t) \\ & \leq \textstyle\int_{T_j^{\varepsilon}} \frac{\pi_1^*(t, C_j^{\varepsilon}, C'_{-j})}{\left|H(t, C_j^{\varepsilon}, C'_{-j})\right|} d\mu(t) + \int_{T \backslash T_j^{\varepsilon}} \frac{\pi_1^*(t, C_j^{\varepsilon}, C'_{-j})}{\left|H(t, C_j^{\varepsilon}, C'_{-j})\right|} d\mu(t) \\ & = \textstyle\int_T \frac{\pi_1^*(t, C_j^{\varepsilon}, C'_{-j})}{\left|H(t, C_j^{\varepsilon}, C'_{-j})\right|} d\mu(t) \\ & = \Pi_j(C_j^{\varepsilon}, C'_{-j}). \end{aligned}$$

Thus, we have

$$\Pi_j < \Pi_j(C_j^{\varepsilon}, C'_{-j}),$$

and thus, firm j can secure an expected catalog profit greater than Π_j.

Case 2: Suppose that for all $j = 1, 2, \ldots, m$, $\mu(T \backslash T_j^+) = 0$, so that for all j, $\pi_j^*(t, C) \geq 0$ a.e.$[\mu]$. If $\Pi_j < \int_T \pi_j^*(t, C) d\mu(t)$ for some j, then we are back to Case 1 above and thus better-reply security holds. Suppose then that

$$\Pi_j = \int_T \pi_j^*(t, C) d\mu(t) \text{ for all } j.$$

We will show that this implies that $|H(t,C)| = 1$ a.e.$[\mu]$. We have

$$\sum_j \Pi_j = \lim_n \sum_j \Pi_j(C^n)$$

$$= \lim_n \int_T \left(\sum_j \frac{\pi_1^*(t,C^n)}{|H(t,C^n)|}\right) d\mu(t)$$

$$\leq \int_T \limsup_n \left(\max_j \pi_j^*(t,C^n)\right) d\mu(t)$$

$$\leq \int_T \max_j \pi_j^*(t,C) d\mu(t)$$

$$\leq \int_T \sum_j \pi_j^*(t,C) d\mu(t) = \sum_j \Pi_j.$$

Since $\pi_j^*(t,C) \geq 0$ a.e.$[\mu]$, $\sum_j \pi_j^*(t,C) \geq \max_j \pi_j^*(t,C)$ a.e.$[\mu]$. Thus, the equality,

$$\int_T \max_j \pi_j^*(t,C) d\mu(t) = \int_T \sum_j \pi_j^*(t,C) d\mu(t),$$

implies that $\sum_j \pi_j^*(t,C) = \max_j \pi_j^*(t,C)$ a.e.$[\mu]$. This in turn implies that either $|H(t,C)| = 1$ a.e.$[\mu]$ or $\sum_j \pi_j^*(t,C) = \max_j \pi_j^*(t,C) = 0$ a.e.$[\mu]$. In either case,

$$\frac{\pi_1^*(t,C)}{|H(t,C)|} = \pi_1^*(t,C) \text{ a.e.}[\mu].$$

Finally, since catalog profile C is not a Nash equilibrium, there exists a firm j and a catalog C'_j such that

$$\Pi_j(C'_j, C_{-j}) > \Pi_j(C_j, C_{-j}) = \Pi_j^*(C_j, C_{-j}) = \Pi_j,$$

and we are back to Case 1 above. ■

From the proof of Proposition 3 above, we can conclude that for any catalog profile $(C_1, \ldots, C_m)$ and any firm j

$$\Pi_j^*(C_j, C_{-j}) - \varepsilon \leq \Pi_j(C_j^\varepsilon, C'_{-j}).$$

for

$$C_j^\varepsilon := \{(x, p - \varepsilon) : (x,p) \in C_j \text{ and } (p - c_j(x)) \geq \varepsilon\},$$

and catalog deviations C'_{-j} by firms other than j such that for each $j' \neq j$, $h(C'_{j'}, C_{j'}) < \delta$. We shall use this fact in the proof of our main result.

4.4 Proof of Main Result: Mixed Catalog Games Are Better-Reply Secure

Let $\{\lambda^n\}_n$ be a sequence of mixed strategies converging to a mixed strategy

$$\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*) \in \Delta(P_f(K_1)) \times \cdots \times \Delta(P_f(K_m)).$$

that is not a Nash equilibrium and suppose that

$$F_j(\lambda^n) \to F_j \text{ for all } j.$$

We have for each j

$F_j = \lim_n \int_{\mathbf{P}} \left[\int_T \frac{\pi_j^*(t,C)}{|H(t,C)|} d\mu(t) \right] d\lambda^n(C)$

$\leq \lim_n \int_{\mathbf{P}} \left[\int_T \pi_j^*(t,C) d\mu(t) \right] d\lambda^n(C)$

$= \lim_n \int_T \left[\int_{\mathbf{P}} \pi_j^*(t,C) d\lambda^n(C) \right] d\mu(t)$

$\leq \int_T \left[\limsup_n \int_{\mathbf{P}} \pi_j^*(t,C) d\lambda^n(C) \right] d\mu(t)$

$\leq \int_T \left[\int_{\mathbf{P}} \pi_j^*(t,C) d\lambda^*(C) \right] d\mu(t) = \int_{\mathbf{P}} \left[\int_T \pi_j^*(t,C) d\mu(t) \right] d\lambda^*(C) = \int_{\mathbf{P}} \Pi_j^*(C) d\lambda^*(C).$

We have two cases:

Case 1: For some firm j, $F_j < \int_{\mathbf{P}} \Pi_1^*(C) d\lambda^*(C)$. Then there is a catalog, C'_j such that

$$F_j < \int_{\mathbf{P}_{-j}} \Pi_1^*(C'_j, C_{-j}) d\lambda_{-j}^*(C).$$

From the proof of Proposition 3, we know that we can choose $C_j'^{\varepsilon}$ so that

$$\Pi_1^*(C'_j, C_{-j}) - \varepsilon \leq \Pi_1(C_j'^{\varepsilon}, C_{-j}) \text{ for all } C_{-j} \in \mathbf{P}_{-j},$$

and we can choose $\varepsilon > 0$ so that

$$F_j < \int_{\mathbf{P}_{-j}} \left(\Pi_1^*(C'_j, C_{-j}) - \varepsilon \right) d\lambda_{-j}^*(C).$$

Thus, we have

$$F_j < \int_{\mathbf{P}_{-j}} \Pi_1(C_j'^{\varepsilon}, C_{-j}) d\lambda_{-j}^*(C).$$

By the payoff security on mixed catalog games (Proposition 2), firm j can secure a payoff greater than F_j. Hence, for case 1, better-reply security holds.

Case 2: For all firms j, $F_j = \int_{\mathbf{P}} \Pi_1^*(C) d\lambda^*(C)$. Let σ^* be the product measure $\mu \times \lambda^*$ and let σ^n be the product measure $\mu \times \lambda^n$. Since λ^n converges to λ^*, σ^n converges to σ^*. Also, let

$$H_{+1} = \{(t,C) \in T \times \mathbf{P} : |H(t,C)| > 1\},$$

and

$$H = T \times \mathbf{P}.$$

We have

$$\int_{\mathbf{P}} \Pi_j^*(C) d\lambda^*(C) = \int_{\mathbf{P}} \left[\int_T \pi_j^*(t,C) d\mu(t) \right] d\lambda^*(C) = \int_H \pi_j^*(t,C) d\sigma^*(C)$$

For case 2 we have

$$\left.\begin{array}{l}\sum_j F_j = \int_{H\backslash H_{+1}} \sum_j \pi_j^*(t,C)d\sigma^*(C) + \int_{H_{+1}} \sum_j \pi_j^*(t,C)d\sigma^*(C) \\ \geq \int_{H\backslash H_{+1}} \sum_j \pi_j^*(t,C)d\sigma^*(C) + \int_{H_{+1}} \max_j \pi_j^*(t,C)d\sigma^*(C) \\ \geq \limsup_n \left[\int_{H\backslash H_{+1}} \sum_j \pi_j^*(t,C)d\sigma^n(C) + \int_{H_{+1}} \max_j \pi_j^*(t,C)d\sigma^n(C)\right] \\ \geq \limsup_n \left[\int_{H\backslash H_{+1}} \sum_j \frac{\pi_j^*(t,C)}{|H(t,C)|}d\sigma^n(C) + \int_{H_{+1}} \sum_j \frac{\pi_j^*(t,C)}{|H(t,C)|}d\sigma^n(C)\right] = \sum_j F_j.\end{array}\right\} (***)$$

Inequalities (***) imply that

$$\int_{H_{+1}} \sum_j \pi_j^*(t,C)d\sigma^*(C) = \int_{H_{+1}} \max_j \pi_j^*(t,C)d\sigma^*(C).$$

Since $\pi_j^*(t,C) \geq 0$ for all $(t,C) \in H_{+1}$ (see expression (19), $\sum_j \pi_j^*(t,C) \geq \max_j \pi_j^*(t,C)$. Thus,

$$\int_{H_{+1}} \sum_j \pi_j^*(t,C)d\sigma^*(C) = \int_{H_{+1}} \max_j \pi_j^*(t,C)d\sigma^*(C)$$

implies that

$$\sum_j \pi_j^*(t,C) = \max_j \pi_j^*(t,C) \text{ a.e.}[\sigma^*] \text{ on } H_{+1}.$$

Thus, either $|H(t,C)| = 1$ a.e.$[\sigma^*]$ or $\sum_j \pi_j^*(t,C) = \max_j \pi_j^*(t,C) = 0$ a.e.$[\sigma^*]$ on H_{+1}. In either case,

$$\frac{\pi_1^*(t,C)}{|H(t,C)|} = \pi_1^*(t,C) \text{ a.e.}[\sigma^*].$$

or$\sum_j \pi_j^*(t,C) = \max_j \pi_j^*(t,C) = 0$ a.e.$[\sigma^*]$ on H_{+1}. Thus, we have

$$\frac{\pi_1^*(t,C)}{|H(t,C)|} = \pi_1^*(t,C) \text{ a.e.}[\mu].$$

We have, therefore, for all j,

$$F_j = \lim_n \int_{\mathbf{P}} \left[\int_T \frac{\pi_j^*(t,C)}{|H(t,C)|}d\mu(t)\right] d\lambda^n(C)$$

$$= \int_{\mathbf{P}} \left[\int_T \pi_j^*(t,C)d\mu(t)\right] d\lambda^*(C)$$

$$= \int_{\mathbf{P}} \left[\int_T \frac{\pi_j^*(t,C)}{|H(t,C)|}d\mu(t)\right] d\lambda^*(C) = F_j(\lambda^*).$$

Since λ^* is not Nash, for some firm j', there exists a mixed strategy, $\lambda'_{j'}$ such that

$$F_{j'}(\lambda'_{j'}, \lambda^*_{-j'}) > F_{j'}(\lambda^*).$$

Choose $\varepsilon > 0$ so that

$$F_{j'}(\lambda'_{j'}, \lambda^*_{-j'}) - \varepsilon > F_{j'}(\lambda^*).$$

By Proposition 3 (payoff security in mixed strategies), firm j' can secure the payoff $F_{j'}(\lambda'_{j'}, \lambda^*_{-j'}) - \varepsilon$. Thus, better-reply security holds for case 2. ■

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