Dynamic Hedging with Stochastic Differential Utility[1]

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Abstract

In this paper we study the dynamic hedging problem using three different utility specifications: stochastic differential utility, terminal wealth utility, and we propose a particular utility transformation connecting both previous approaches. In all cases, we assume Markovian prices. Stochastic differential utility, SDU, impacts the pure hedging demand ambiguously, but decreases the pure speculative demand, because risk aversion increases. We also show that consumption decision is, in some sense, independent of hedging decision. With terminal wealth utility, we derive a general and compact hedging formula, which nests as special all cases studied in Duffie and Jackson (1990). We then show how to obtain their formulas. With the third approach we find a compact formula for hedging, which makes the second-type utility framework a particular case, and show that the pure hedging demand is not impacted by this specification. In addition, with CRRA- and CARA-type utilities, the risk aversion increases and, consequently the pure speculative demand decreases. If futures price are martingales, then the transformation plays no role in determining the hedging allocation. We also derive the relevant Bellman equation for each case, using semigroup techniques.

Keywords*:* Stochastic Control, Recursive Utility, Hedging, Bellman Equation

JEL Classification: C61, D92, G11

1 INTRODUCTION

In this paper we aim at studying the optimal futures hedging problem using a continuous-time setting, stochastic differential utility (Duffie and Epstein, 1992), SDU, and assuming Markovian prices as in Adler and Detemple (1988). Then we extend the model in Duffie and Jackson (1990) - henceforth referred to as DJ - in three ways. In the first, we maximize the intertemporal consumption, in the spirit of Ho (1984). The consumption approach is interesting because in an intertemporal context, the agent is willing to stabilize the expected consumption stream. In order to perform such a task, the agent adjusts present consumption to account for investment for future consumption and holds positions in futures. The latter is called futures hedging and the former consumption hedging. Recursive utility disentangles risk aversion from intertemporal substitutability, and therefore it may allow us to enhance the efficiency of the hedging strategy by means of its less restrictive structure. Consequently, we think of this approach as a tool which hedgers may use in order to improve the performance of their futures position, in such a way that SDU adds some degrees of freedom in a possible empirical treatment still to be undertaken. However, with standard SDU, the formula that we obtain depends upon the derivatives of the value function, which is not easy to figure out. Thereby, to solve the optimal hedge, we potentially have to employ numerical methods.

Afterwards, in the second case, the agent maximizes the terminal wealth, a usual assumption when we are only worried about a specific future date. Maximizing the utility of the terminal wealth has an advantage of producing closed formulas for some cases that arise naturally, and, hence, our analysis may be deepened further. As a matter of fact, we are able to produce a general hedging formula, which nests as special all cases studied in DJ's paper. Then we specialize it to compare with those obtained by them.

As a consequence of regarding prices as Markov processes, the myopic hedging problem at each time[1] no longer holds necessarily. Hence, our optimal futures hedging formulas are not the same as the corresponding static hedges, nor directly comparable to analogous solutions in discrete-time cases, as the results in Anderson and Danthine (1981, 1983).

We would like to find some link between both approaches that we have mentioned. We do this by making a suitable transformation on the Terminal Value-type utility. Then we take advantage of the compact formulas that it produces and see explicitly what effect the SDU specification exerts on the standard model, that is, we introduce the certainty equivalent machinery in the Hamilton-Jacobi-Bellman equation, HJB, of the terminal value utility and study what effects this causes on the optimal hedge ratio. The particular concave transformation that we employ is the connection between SDU and Terminal Utility. This adds some degrees of freedom and may potentially improve the hedging strategy. The main advantage of this specification, hence, is to make DJ's model a special case, and to allow us to derive interesting and neat results.

Our model is similar to DJ in several aspects: spot and futures prices are vector diffusion processes; a hedge is a vector stochastic process which specifies a futures position in each futures market; and the hedging profits

[1]Myopic hedging at each time means that an agent is hedging only local changes in wealth. Further discussion can be found in Adler and Detemple (1988) and the references therein.

and losses are marked to market in an interest-paying margin account. The optimal hedge ratio will be obtained by maximizing the expected utility, composed by a committed portfolio of spot markets and the accumulated value of the margin account.

We reserve the appendix to derive the relevant HJB equations, taking advantage of the semigroup approach.

The paper is organized as follows: in Section 2, we present the primitives of our model; in Section 3, we derive the optimal hedge under SDU; in Section 4, we derive the optimal hedge ratio maximizing only the terminal utility and make some specializations to compare our results with some of those obtained by DJ and also previously in the paper; in Section 5, we derive the optimal hedge ratio under the proposed utility transformation and compare with our previous results; then, in Section 6, we conclude. In the appendix, we discuss how to obtain the relevant HJB equations; we provide some explanations about the differential utility, and also about some mathematical concepts that we use to derive the optimal hedge.

2 THE MODEL

As mentioned earlier, our model is closely similar to DJ's, because we consider an agent choosing a future trading strategy to maximize the expected utility of consumption from t to a future time $T \in \mathbb{R}_+ \cup \infty$, according to the following setup.

1. Let $B = \left(B^1, B^2, \ldots, B^N\right)'$ denote a Standard Brownian Motion in $\mathbb{R}^N$ which is a martingale with respect to the agent's filtered proba-

bility space[2]. We also assume throughout the paper that probabilistic statements are in the context of this filtered probability space.

2. V denotes the space of predictable square-integrable processes[3], such that

$$V \equiv \left\{ \text{predictable } \upsilon : [0,T] \times \Omega \to \mathbb{R} \middle| E\left[\int_0^t \upsilon_s^2 ds\right] < \infty, t \in [0,T] \right\},$$

where Ω is the state space, and predictable means measurable with respect to the σ-algebra generated by left-continuous processes adapted to the agent's filtration, that is, υ_t depends only on information available up to time t.

3. There exist M assets to be hedged, whose value is described by an M-dimensional Markov process S, with the stochastic differential representation

$$dS_t = \mu_t(S_t)dt + \sigma_t(S_t)dB_t, \tag{1}$$

where μ is M-dimensional, σ is $(M \times N)$-dimensional and $\mu^m \in V$ and $\sigma^{mn} \in V$ for all m and n (hence, the Markov process S is well defined)[4].

4. There are K futures contracts available for trade at each instant of

[2] "$'$" indicates transpose, or differentiation when there is only one argument in the function.

[3] If $T = \infty$, then the square integrability condition changes to $E\left[\int_0^\infty e^{\beta t}\upsilon_s^2 ds\right] < \infty$, where β is a constant characterized in Appendix C of Duffie and Epstein (1992).

[4] Henceforth, we omit the dependence of the parameters on S_t for simplicity.

time, whose prices are given by a K-dimensional Ito process F with the stochastic differential representation

$$dF_t = m_t(F_t)dt + \upsilon_t(F_t)dB_t, \tag{2}$$

where $m^k \in V$ and $\upsilon^{kn} \in V$ for all k and n[5].

5. A futures position is taken by marking to market a margin account according to a K-dimensional process $\theta = \left(\theta^1, \theta^2, \ldots, \theta^K\right)'$, with the property that $\theta' m$ as well as each element of $\theta' \upsilon$ belong to V. The space Θ of all such futures position strategies is then described by

$$\Theta = \left\{\theta \left| \theta' m \in V \text{ and } \theta' \upsilon^n \in V, \forall n \right.\right\}.$$

6. At time t the position θ_t in the K contracts is credited with any gains or losses incurred by futures price changes, and the credits (or debits) are added to the agent's margin account. Thus, the margin account's current value, denoted by X_t^θ, is credited with interest at the constant continuously compounding rate $r \geq 0$. Furthermore, we assume that losses bringing the account to a negative level are covered by borrowing at the same interest rate, and ignore transactions costs and other institutional features. In a continuous-time model, the margin account then has the form

$$X_t^\theta = \int_0^t e^{r(t-s)} \theta_s' dF_s, \tag{3}$$

[5] Idem with respect to F_t.

indicating that the 'increment' $\theta_s' dF_s$ to the margin account at time s is re-invested at the rate r, implying a corresponding increment of $e^{r(t-s)}\theta_s' dF_s$ to the margin account by time t. Its equivalent stochastic differential equation applying Ito's Lemma is

$$dX_t^\theta = \left(rX_t^\theta + \theta_t' m_t\right) dt + \theta_t' \upsilon_t dB_t. \tag{4}$$

7. Let $\pi_t \in \mathbb{R}^M$ be a bounded measurable function standing for the agent spot commitment. For simplicity, we also assume that the agent does not invest in any risky asset. Hence, the total wealth of the agent at time t, given a futures position strategy θ, is then W_t^θ, where W^θ is the Ito process having the stochastic differential representation

$$dW_t^\theta = \pi_t' dS_t + dX_t^\theta - c_t dt, \tag{5}$$

where $c_t \in V$ is the consumption rate at time t.

8. Preferences of the agent over wealth at time t are given by the stochastic differential utility[6] $U : V \to \mathbb{R}$, whose "aggregator", (f, k), is defined as $f : \mathbb{R}^D \times \mathbb{R} \to \mathbb{R}$, and the variance multiplier, k, as $k : \mathbb{R} \to \mathbb{R}$. We define f to be regular, meaning that f is continuous, Lipschitz in utility, and satisfies a growth condition in consumption[7]. In addition, we assume that f is increasing and concave in consumption[8]. Consider

[6] More details about SDU are given in the appendix B

[7] More details about the meaning of these concepts, see DJ and Duffie and Epstein (1992, p. 366).

[8] Then, we can apply freely Propositions 3 (monotonicity) and 5 (concavity) of Duffie and Epstein (1992).

the von Newmann-Morgenstern index, h, which is continuous, strictly increasing, and satisfies a growth condition, implying it is integrable. It is called the risk-adjustment function and measures the local risk aversion. Because we also adopt assumption 2 in Duffie and Epstein (1992), we obtain $k(J) = \frac{h''(J)}{h'(J)} < 0$. This leaves the problem

$$\max_{\theta \in \Theta, c \in V} E_t \int_{s \geq t}^{T} \left[f\left(c_s, J\left(z_s\right)\right) + \frac{1}{2} k\left(J\left(z_s\right)\right) J'_{z_s} \Sigma J_{z_s} \right] ds, \tag{6}$$

with $\delta > 0$ being the subjective discount rate[9].

With this model in mind, we can define the optimal futures position.

Definition 1 *A futures position strategy θ is defined to be optimal if it solves 6.*

3 STOCHASTIC DIFFERENTIAL UTILITY

In this section we derive directly the optimal futures hedging, under Markovian prices and stochastic differential utility.

Proposition 1 *The optimal futures position strategy is θ^{SDU}, where*

$$\begin{aligned} \theta_t^{SDU} &= -\frac{\left(J_{ww} + J_{wx}\right) + k(J)\left(J_w^2 + J_w J_x\right)}{\left(J_{ww} + 2J_{wx} + J_{xx}\right) + k(J)\left(J_w^2 + 2J_w J_x + J_x^2\right)} \times \\ &\quad \left(\upsilon_t \upsilon_t'\right)^{-1} \left[\upsilon_t \sigma_t' \pi_t + \frac{J_w + J_x}{\left(J_{ww} + J_{wx}\right) + k(J)\left(J_w^2 + J_w J_x\right)} m_t \right]. \end{aligned} \tag{7}$$

[9] Without recursive utility, $f\left(c_s, J\left(z_s\right)\right) = u\left(c_s\right) - \delta J\left(z_s\right)$, and $k(J) = 0$.

Proof. *Let $W^{\theta t}$ define the wealth process that would obtain starting at time t and with futures strategy θ, translating time parameters back t time units to time 0, or*

$$dW_s^{\theta t} = a_{t+s}ds + b'_{t+s}dB_s,$$

where $a_{t+s} = rX_s^{\theta t} + \pi'_t\mu_{t+s} + \theta'_{t+s}m_{t+s} - c_{t+s}$, and $b_{t+s} = \sigma'_{t+s}\pi_t + \upsilon'_{t+s}\theta_{t+s}$. Similarly, let $X^{\theta t}$ be the t-translate of the process X defined by

$$dX_s^{\theta t} = \alpha_{t+s}ds + \beta'_{t+s}dB_s,$$

where $\alpha_{t+s} = rX_s^{\theta t} + \theta'_{t+s}m_{t+s}$, and $\beta_{t+s} = \upsilon'_{t+s}\theta_{t+s}$.

Define the value function $J : \mathbb{R}^2 \to \mathbb{R}$ by

$$J(z_t) = \max_{\theta\in\Theta, c\in V} E_t \int_{s\geq t}^{T} \left[f(c_s, J(z_s)) + \frac{1}{2}k(J(z_s))\, J'_{z_s}\Sigma J_{z_s} \right] ds,$$

and let $\widehat{J} : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\widehat{J}(z_t) = E_t[U(c)]$, where $z = (w, x)$, $Z_0^{\theta^{SDU}t} \equiv \left(W_0^{\theta^{SDU}t}, X_0^{\theta^{SDU}t}\right) = (w, x)$, with the boundary condition as $\widehat{J}(z_T) = 0$[10].

Notice that:

$$dZ_t = \mu_{z_t}dt + \Lambda_{z_t}dB_t,$$

where $\mu_{z_t} = (a_t, \alpha_t)'$, and $\Lambda_{z_t} = (b_t, \beta_t)'$.

The HJB equation with recursive utility is given by $u(c_t) - \delta J + \mathcal{A}_d J +$

[10] *In special situations, we may vary from the convention of having zero terminal utility (see Duffie and Epstein, 1992).*

$\frac{1}{2}k(J)\left(J'_{z_t}\Sigma J_{z_t}\right) = 0$, *where $\mathcal{A}_d$ is the infinitesimal generator*[11].

Then, using the HJB equation we get:

$$\begin{aligned}
\delta J &= \sup_{\theta\in\mathbb{R}^k, c\in V} \left\{ u(c_t) + J_w a_t + J_x \alpha_t + \right. \\
&\left. + \frac{1}{2} tr\left[J_{ww} b_t b'_t + 2J_{wx}\beta_t b'_t + J_{xx}\beta_t \beta'_t + k(J)\Sigma J_{z_t} J'_{z_t} \right] \right\}.
\end{aligned}$$

Two observations are in order here. First, this program can be split into two independent decisions:

$$\sup_{c\in V} \left(u(c_t) - J_w c_t \right), \text{ and}$$

$$\begin{aligned}
&\sup_{\theta\in\mathbb{R}^k} \left\{ J_w \left(rX_s^{\theta t} + \pi'_t \mu_{t+s} + \theta'_{t+s} m_{t+s} \right) + J_x \alpha_t + \right. \\
&\left. + \frac{1}{2} tr\left[J_{ww} b_t b'_t + 2J_{wx}\beta_t b'_t + J_{xx}\beta_t \beta'_t + k(J)\Sigma J_{z_t} J'_{z_t} \right] \right\}.
\end{aligned}$$

Second, notice that:

a. $b'_t b_t = \left(\sigma'_t \pi_t + \upsilon'_t \theta_t\right)' \left(\sigma'_t \pi_t + \upsilon'_t \theta_t\right) = \pi'_t \sigma_t \sigma'_t \pi_t + 2\pi'_t \sigma_t \upsilon'_t \theta_t + \theta'_t \upsilon_t \upsilon'_t \theta_t \Rightarrow \frac{\partial b'_t b_t}{\partial \theta_t} = 2\upsilon_t \sigma'_t \pi_t + 2\upsilon_t \upsilon'_t \theta_t$;

b. $b'_t \beta_t = \left(\sigma'_t \pi + \upsilon'_t \theta_t\right)' \upsilon'_t \theta_t = \pi'_t \sigma_t \upsilon'_t \theta_t + \theta'_t \upsilon_t \upsilon'_t \theta_t \Rightarrow \frac{\partial b'_t \beta_t}{\partial \theta_t} = \upsilon_t \sigma'_t \pi_t + 2\upsilon_t \upsilon'_t \theta_t$;

c. $\beta'_t \beta_t = \theta'_t \upsilon_t \upsilon'_t \theta_t \Rightarrow \frac{\partial \beta'_t \beta_t}{\partial \theta_t} = 2\upsilon_t \upsilon'_t \theta_t$.

[11] *See appendix for details.*

Then it follows from the first order conditions for θ that:

$$\begin{aligned} 0 &= (J_w + J_x)\, m_t + \left(J_{ww} + k(J) J_w^2\right)(\upsilon_t \sigma_t' \pi_t + \upsilon_t \upsilon_t' \theta_t) + \\ &\quad (J_{wx} + k(J) J_x J_w)(\upsilon_t \sigma_t' \pi_t + 2\upsilon_t \upsilon_t' \theta_t) + \left(J_{xx} + k(J) J_x^2\right) \upsilon_t \upsilon_t' \theta_t. \end{aligned}$$

Collect terms, and the result follows. ■

The characterization of the optimal strategy is in terms of the derivatives of the value function as in Breeden (1984), Ho (1984), and Adler and De-Temple (1988); thereby, potentially we may have to solve this equation by numerical methods. We can see that the optimal consumption is determined independently of the hedging decisions. This is a result similar to Ho (1984). Observe that the drift of the spot prices does not affect explicitly the futures optimal hedging. It may affect through other channels, however, as for instance, the derivatives of the Bellman equation. The utility parameters, the volatility parameters of the prices and the drift of the futures prices affect the optimal strategy in an evident way. Of course, it is easy to notice that if $k(J) = 0$, then we have optimal futures hedging formula for the standard additive utility specification[12].

[12] See also the hedging formula in Section 4. This happens because of the independence between consumption and hedging decision.

Under our assumptions about the utility and the aggregator, at the optimal, the first multiplicative term is positive[13]:

$$A_t \equiv \frac{\left(\overbrace{J_{ww}+J_{wx}}^{-}\right)+\overbrace{k(J)}^{-}\left(\overbrace{J_w^2+J_wJ_x}^{+}\right)}{\left(\underbrace{J_{ww}+2J_{wx}+J_{xx}}_{-}\right)+k(J)\left(\underbrace{J_w^2+2J_wJ_x+J_x^2}_{+}\right)} > 0.$$

Given we do not know the functional form the value function, it is difficult to say if A is greater or less than that under standard utility. However, we can say that its sign does not change under SDU.

In addition, the term that multiplies m_t is negative:

$$-R_t^{-1} \equiv \frac{\overbrace{J_w+J_x}^{+}}{\left(\underbrace{J_{ww}+J_{wx}}_{-}\right)+\underbrace{k(J)}_{-}\left(\underbrace{J_w^2+J_wJ_x}_{+}\right)} < 0.$$

We interpret R as being an extended version of the risk aversion coefficient because it includes terms like J_x and J_{wx}. The numerator is a measure of the global concavity of the value function plus the term originated from the SDU approach; and the denominator stands for the global curvature of the value function[14]. If the value function becomes more concave, either because

[13] From the first order conditions $0 < J_w = u'(c)$, and because c is a normal good, such that $\frac{\partial c}{\partial w}, \frac{\partial c}{\partial x} > 0$.

[14] With Terminal Utility, as we will see later, this term becomes the standard coefficient of risk aversion $-\frac{J_{ww}}{J_w}$.

of the wealth or because the margin account, R increases as in the standard case. Furthermore, notice that the SDU utility adds a penalty through the presence of the additional term in the numerator: If the curvature of the value function increases, both numerator and denominator increase, making the net effect unknown (in the standard case, risk aversion would decrease.) The sign of R does not change under SDU. In short, the global net effect is that the extended risk aversion increases because of the presence of the mentioned term in the denominator. This is, of course, an implication from Proposition 6 (Comparative Risk Aversion) in Duffie and Epstein (1992).

With this in mind, Duffie (1989) arguably calls the first term between brackets in equation 7 as the *pure hedge demand*, and the second term as the *pure speculative demand*[15]. The term *pure hedge demand* comes from a uniperiod model, where we want only to minimize risks, that is, we minimize the variance of our position, without preoccupations with the return. In this case, $\theta_t = (\upsilon_t \upsilon_t')^{-1} \upsilon_t \sigma_t' \pi_t$ (see additional discussion in Section 4.1).

If the spot commitment is zero at time t, the hedger may still be willing to buy futures through the pure speculative demand term. This also would be so if covariance between spot and futures prices were null - υ_t is orthogonal to σ_t -, meaning that futures contracts do not provide any protection against spot price fluctuations[16]. Also, if the covariance between futures and spot prices increases in absolute value, one increases the position under hedge, since the role of protecting against undesirable fluctuation in prices increases.

[15] Adler and Detemple (1988) call them, respectively, as the Merton/Breeden informationally based hedging component, and as the mean-variance component.

[16] Here we are not taking into account equilibrium concerns.

Looking at the second part inside the brackets, the formula shows that the multiplicative term of m_t has increased with the extra term in the denominator, for the entire expression is negative as we have already shown. Thus, its absolute value decreases. Therefore the contribution of the pure speculative demand to the optimal hedge strategy decreases with recursive utility. If $m_t = 0$, however, the solution does not depend explicitly on the drift of futures prices, but differently from DJ, it depends on the parameters of the recursive utility through $k(J)$ and the other derivatives.

4 TERMINAL WEALTH

In this section, we practically hold the same model as before. However, we make a fundamental modification by maximizing the utility of the terminal wealth, instead of the consumption over time. As a consequence, we consider here a finite time, T. We also make a simplifying assumption by considering the spot commitment constant over time. This makes our model identical to DJ's, except that our prices are Markovian. This assumption about the prices is not new; Adler and Detemple (1988) have assumed it in a model constructed to solve a similar problem. Formally, we have:

1. The agent is committed to receiving the value at time T of a position in these assets represented by a fixed portfolio $\pi \in \mathbb{R}^M$, leaving the terminal value $\pi' S_T$. Hence, the total wealth of the agent at time T, given a futures position strategy θ, is then W_T^θ, where W^θ is the Ito

process having the stochastic differential representation

$$dW_t^\theta = \pi' dS_t + dX_t^\theta. \tag{8}$$

2. Preferences of the agent over wealth at time T are given by von Newman-Morgenstern utility function $U : \mathbb{R} \to \mathbb{R}$, which is monotonic, twice continuously differentiable, strictly concave, with U' and U'' each satisfying a (linear) growth condition. This leaves the problem

$$\max_{\theta \in \Theta} E_t \left[U \left(W_T^\theta \right) \right]. \tag{9}$$

Of course the optimal position now is the one that maximizes statement 9. Then we can state the proposition:

Proposition 2 *The optimal futures position strategy, by maximizing the terminal utility, is θ^{TV}, where*

$$\theta_t^{TV} = -\frac{(J_{ww} + J_{wx})}{(J_{ww} + 2J_{wx} + J_{xx})} \left(\upsilon_t \upsilon_t' \right)^{-1} \left[\upsilon_t \sigma_t' \pi_t + \frac{J_w + J_x}{(J_{ww} + J_{wx})} m_t \right] \tag{10}$$

Proof. *The proof is the same as the one presented in the last section except that $\pi_{t+s} = \pi$, $c_{t+s} = 0, \forall\, s, t \geq 0$, and we define the value function $J : \mathbb{R}^2 \to \mathbb{R}$ by*

$$J(z_t) = \max_{\theta \in \Theta} E \left[U \left(W_{T-t}^{\theta t} \right) \right].$$

Since we are maximizing as expected utility at a future date T, the HJB *equation of this case is a little bit different, and it is given by* $\mathcal{A}_d J = 0$[17].

In this case, we define the boundary condition as $\widehat{J}\,(z_T) = U(w)$. ■

Notice that we are only worried about the local mean effect, because of our interest in the terminal value of the wealth at the maturity of the contract. Also, observe the similarity of this expression with that obtained under SDU, had we assumed $k(J) = 0$.

The qualitative analysis we provided before regarding the formula under SDU holds here without modification, thus we do not repeat it.

The formula can be compacted further! In order to do that, let us state an important result of our model:

Proposition 3 *The net return of the variation in the final wealth given a variation in the initial wealth is equal to the variation in the final wealth given a variation in the initial margin account. Formally,*

$$\{\exp\left[r\left(T-t\right)\right]-1\}\frac{\partial W_{T-t}^{\theta t}}{\partial w}=\frac{\partial W_{T-t}^{\theta t}}{\partial x}.$$

Proof. *Itô's lemma on equation 4 gives us:*

$$X_t^{\theta}=X_0^{\theta}+\int_0^t\left(rX_s^{\theta}+\theta_s'm_s\right)ds+\int_0^t\theta_s'\upsilon_s dB_s.$$

Differentiate X_t^{θ} *with respect to* X_0^{θ}*:*

$$\frac{\partial X_t^{\theta}}{\partial X_0^{\theta}}=1+r\int_0^t\frac{\partial X_s^{\theta}}{\partial X_0^{\theta}}ds.$$

[17] *See appendix C or Krylov (1980) for a proof.*

Define $y_s \equiv \frac{\partial X_s^\theta}{\partial X_0^\theta}$. Then the expression is an ordinary differential equation, whose solution is given by

$$y_t = y_0 e^{rt}, y_0 = 1.$$

Hence

$$\frac{\partial X_{T-t}^\theta}{\partial X_0^\theta} = e^{r(T-t)}.$$

By Itô's lemma again on equation 8:

$$\begin{aligned} W_{T-t}^{\theta t} &= W_0^{\theta t} + \int_0^{T-t} a_{t+s} ds + \int_0^{T-t} b'_{t+s} dB_s = \\ &= W_0^{\theta t} + \int_0^{T-t} \pi' \mu_{t+s} ds + \int_0^{T-t} \left(\sigma'_{t+s}\pi\right)' dB_s + X_{T-t}^{\theta t} - X_0^{\theta t} = \\ &= W_0^{\theta t} + \pi' \left(S_{T-t} - S_0\right) + X_{T-t}^{\theta t} - X_0^{\theta t}. \end{aligned}$$

Consequently the result follows. ■

Proposition 3 says that a positive variation in the initial wealth implies a positive variation in the final wealth, and hence, in the value function, as we will see soon. Similar argument can be stated with respect to the initial position in the margin account. In the proof, notice that an increase in the initial margin account amounts to a gross return growth in the final margin during the period considered.

This result does not rely on the utility assumptions, but only on the budget constraint. Consequently this result holds whether we are at the optimal point or not.

The next statement is a direct consequence from this proposition. It makes a connection between the derivatives of the value function:

Corollary 1 *Given the assumptions about the utility and budget constraint, then the following equality holds:*

$$\left\{ \exp\left[r\left(T-t\right) \right] -1\right\} J_{w}=J_{x}.$$

Proof. *First observe that*

$$J(z_{t})=E\left[U\left(W_{T-t}^{\theta t}\right) \right]$$

Differentiate inside the expectation[18]*:*

$$\begin{aligned} 0 &< \frac{\partial J}{\partial w}=E\left[U^{\prime }\left(W_{T-t}^{\theta t}\right) \frac{\partial W_{T-t}^{\theta t}}{\partial w}\right] ; \\ 0 &< \frac{\partial J}{\partial x}=E\left[U^{\prime }\left(W_{T-t}^{\theta t}\right) \frac{\partial W_{T-t}^{\theta t}}{\partial x}\right] . \end{aligned}$$

Apply Proposition 3 and the result follows. ■

From this Corollary 1, it arises another interesting corollary that is stated now:

Corollary 2 *The following equality holds:*

$$\left\{ \exp\left[r\left(T-t\right) \right] -1\right\} ^{2}J_{ww}=\left\{ \exp\left[r\left(T-t\right) \right] -1\right\} J_{wx}=J_{xx}.$$

[18] *See appendix in DJ, which states the sufficient conditions for differentiation inside the expectation. These conditions are met here, because of our assumptions on the utility function.*

Proof. *Differentiate inside the expectation again:*

$$\begin{aligned}
\frac{\partial^2 J}{\partial w^2} &= E\left[U''\left(W_{T-t}^{\theta t}\right)\left(\frac{\partial W_{T-t}^{\theta t}}{\partial w}\right)^2 + U'\left(W_{T-t}^{\theta t}\right)\frac{\partial^2 W_{T-t}^{\theta t}}{\partial w^2}\right]; \\
\frac{\partial^2 J}{\partial x^2} &= E\left[U''\left(W_{T-t}^{\theta t}\right)\left(\frac{\partial W_{T-t}^{\theta t}}{\partial x}\right)^2 + U'\left(W_{T-t}^{\theta t}\right)\frac{\partial^2 W_{T-t}^{\theta t}}{\partial x^2}\right]; \\
\frac{\partial^2 J}{\partial w \partial x} &= E\left[U''\left(W_{T-t}^{\theta t}\right)\frac{\partial W_{T-t}^{\theta t}}{\partial w}\frac{\partial W_{T-t}^{\theta t}}{\partial x} + U'\left(W_{T-t}^{\theta t}\right)\frac{\partial^2 W_{T-t}^{\theta t}}{\partial w \partial x}\right].
\end{aligned}$$

Since all the terms that multiply $U'\left(W_{T-t}^{\theta t}\right)$ are zero, the result follows by applying Proposition 3. ■

Observe that all second derivatives are negative, since $U\left(W_{T-t}^{\theta t}\right)$ is assumed concave.

Now if we replace the relationships that we obtained in the optimal hedging equation, we get the following more compact hedge ratio:

Proposition 4 *Given our assumption on the budget constraint and utility, the optimal hedging ratio is given by*

$$\theta_t^{TV} = -\exp\left[-r\left(T-t\right)\right]\left(\upsilon_t \upsilon_t'\right)^{-1}\left[\upsilon_t \sigma_t' \pi + \frac{J_w}{J_{ww}} m_t\right]. \tag{11}$$

Proof. *Replace terms.* ■

The main discovery is that the first and big term outside the brackets reduces to a deterministic expression, which does not depend upon the form of the value function. Also, adding a discount factor in the model would change the HJB equation accordingly, but would not have effects on the optimal hedging.[19]

In order to sharpen further our analysis, we make some simplifying assumptions in the next subsections, and show how our results are related to DJ's.

4.1 MARTINGALES FUTURES PRICES - $m_t = 0$

Our first special case occurs when the drift of the futures price is identically equal to zero. This also happens in 4 out of the 5 cases studied in DJ.

Proposition 5 *Under our assumptions coupled with $m_t = 0$, the optimal futures position strategy is θ^{TV}, where*

$$\theta_t^{TV} = -e^{-r(T-t)} \left(\upsilon_t \upsilon_t' \right)^{-1} \upsilon_t \sigma_t' \pi. \tag{12}$$

This result is very similar to cases 1, 3 and 4 in DJ, where they assume in all of them martingales futures prices. In their case 1, in addition, they assume Gaussian prices; case 3, mean-variance preferences; and case 4 log-normal spot prices and mean-variance utility. Here, we obtain their results under more general assumptions. The bad news is that, if $m_t = 0$, the utility parameters play no role in the formula. The explanation given by DJ is that the demand for futures is based only on the hedge they provide, in such a way that futures are only used to control 'noise' in the portfolio process. Still quoting DJ, "the optimal manner of doing so depends solely on the structure of the 'noise' in the price process, not on the structure of the utility, nor on the drift of the assets' price processes."

[19]In Appendix D, Remark, we discuss more deeply this option.

In particular, let us see what happens if we specify martingale futures price and log-normally distributed spot prices. This is a very special case, and to make it tractable, assume for the m-th spot price:

$$dS_t^m = g_t^m S_t^m dt + S_t^m h_t^{m\prime} dB_t.$$

In addition, to make this process log-normally distributed, we must impose g_t^m and h_t^m deterministic, where $g_t^m : [0,T] \to \mathbb{R}$ and $h_t^m : [0,T] \to \mathbb{R}^N$ are bounded measurable functions.

Proposition 6 *Under martingale futures prices and log-normally distributed spot prices, the optimal hedge ratio is*

$$\theta_t^{TV} = -e^{-r(T-t)} \left(\upsilon_t \upsilon_t'\right)^{-1} \upsilon_t H_t' \pi, \tag{13}$$

where H_t is the $M \times N$ matrix whose m^{th} row is $S_t^m \exp\left[\int_t^T \left(g_s^m - \frac{1}{2} h_s^{m\prime} h_s^m\right) ds\right] h_t^{m\prime}$.

Proof. *See DJ.* ■

4.2 EXPONENTIAL UTILITY - $m_t \neq 0$

Supposing that we have an exponential utility is to assume $u(w) = -e^{-\gamma w}$, where $\gamma > 0$ is a constant measure of risk aversion. As it is well known, this kind of utility is often used in continuous-time investment studies.

Proposition 7 *Under exponential utility, the optimal futures position strategy is θ^*, where*

$$\theta_t^* = -e^{-r(T-t)} \left(\upsilon_t \upsilon_t'\right)^{-1} \left[\upsilon_t \sigma_t' \pi - \frac{1}{\gamma} m_t\right]. \tag{14}$$

Proof. *Observe that*

$$
\begin{aligned}
J_w &= -\gamma E_t \left[U \left(W_{T-t}^{\theta^* t} \right) \frac{\partial W_{T-t}^{\theta^* t}}{\partial w} \right]; \text{ and} \\
J_{ww} &= \gamma E_t \left[\gamma U \left(W_{T-t}^{\theta^* t} \right) \left(\frac{\partial W_{T-t}^{\theta^* t}}{\partial w} \right)^2 - U \left(W_{T-t}^{\theta^* t} \right) \frac{\partial^2 W_{T-t}^{\theta^* t}}{\partial w^2} \right].
\end{aligned}
$$

Since $\frac{\partial W_{T-t}^{\theta^* t}}{\partial w} = 1$, *the result is immediate.* ■

This is the second-case result in DJ holds as expected. Here it is easy to see that if the risk aversion increases, the pure speculative demand on the optimal hedge decreases.

We could consider other commonly used utility functions, such as power and quadratic functions. But, then, we must deal with two problems. First, for example, power functions do not admit negative values for wealth. Second, and more practical, in such cases we would have to use numerical methods to find the optimal hedge, because the expected wealth that would appear at the RHS depends on the optimal hedging strategy.

We conclude this subsection just recalling that a combination of the assumptions derived individually could be done in order to obtain a more complete optimal hedging strategy.

4.3 EQUILIBRIUM

Assume exponential utility, such that $u_i(w) = -e^{-\gamma_i w}$ stands for the agent's Von Newmann-Morgenstern utility for terminal wealth[20]. Then, we obtain:

$$\theta_{it}^* = -e^{-r(T_i-t)} \left(\upsilon_t \upsilon_t'\right)^{-1} \left[\upsilon_t \sigma_t' \pi_{it} - \frac{1}{\gamma_i} m_t\right],$$

where T_i represents the terminal date of agent i.

Market clearing, $\sum_{i=1}^{I} \theta_{it} = 0$, implies that:

$$m_t = \upsilon_t \sigma_t' \frac{\sum_{i=1}^{I} e^{-r(T_i-t)} \pi_{it}}{\sum_{i=1}^{I} \frac{e^{-r(T_i-t)}}{\gamma_i}} = \upsilon_t \sigma_t' \frac{\sum_{i=1}^{I} \omega_{it} \pi_{it}}{\sum_{i=1}^{I} \frac{\omega_{it}}{\gamma_i}}, \qquad (15)$$

where $\omega_{it} \equiv \frac{e^{-r(T_i-t)}}{\sum_{i=1}^{I} e^{-r(T_i-t)}}$.

First, m_t is proportional to each individual spot position, whose weights are given by ω_i. If the covariance is high, that means that the futures contracts provide a good hedge, increasing the demand for hedging. Finally m_t is proportional to the risk aversion of investors. Higher levels of risk aversion correspond to higher m_t, that is, an increasing in the pure speculative demand.

Sufficient conditions for having $m_t = 0$ are: (a) $\sum_{i=1}^{I} \omega_{it} \pi_{it} = 0$, that is, there is no excess demand for hedging; (b) $\upsilon_t \sigma_t' = 0$, in which case the futures provide no hedge; or (c) $\gamma_i = 0$ for some agent i. In case (a) agents can costlessly insure themselves since there exists always someone who desires to take an opposite position, and then speculators are unnecessary in this market.

[20] This section is closely similar to section 4 in DJ.

And substituting this expression in the hedge ratio, we get:

$$\theta_{jt}^{*} = -e^{-r(T_j - t)} \left(\upsilon_t \upsilon_t'\right)^{-1} \upsilon_t \sigma_t' \left[\pi_{jt} - \frac{\sum_{i=1}^{I} \omega_{it}\pi_{it}}{\gamma_j \sum_{i=1}^{I} \frac{\omega_{it}}{\gamma_i}}\right].$$

5 CONNECTING SDU AND TERMINAL WEALTH

We would like to find some link between both approaches that we have been studying. We may do this by making a suitable transformation on the Terminal Value-type utility. We might then take advantage of the compact formulas generated by the second-type utility and explicitly see what effect the SDU model exerts on the optimal hedge ratio. The effect of the SDU on the hedge ratio comes only from the certainty equivalent machinery. For this purpose we make a particular concave transformation of the terminal value-type utility. Formally, then, we have:

1. Preferences of the agent over wealth at time t are given by the utility $U : V \to \mathbb{R}$, defined as [21]

$$\max_{\theta \in \Theta} h^{-1} \left(E_t \left[h \left(U \left(W_T^{\theta t}\right)\right)\right]\right).$$

[21] An alternative, and equivalent, characterization would be to maximize $J(z_t) = E_t \left[U \left(W_T^{\theta t}\right)\right] + \frac{1}{2} E_t \int_{s \geq t}^{T} k \left(J \left(z_s\right)\right) J_{z_s}' \Sigma J_{z_s} ds$ (see appendix). This would generate the same HJB equation, and hence the same evolution of the value function. However, additional derivations that we make later on, if started from this definition, would be messier. This is the reason that we have chosen that definition in the main text.

The practical advantage is to produce neat formulas for our problem. In theoretical grounds we are assuming that we can add the certainty equivalent in the HJB equation when we are maximizing the utility of the final wealth. Our procedure lies in the remarkable consequence of the forward-looking nature of the Bellman equation under SDU, dispensing with state variables reflecting past decisions (see Duffie and Epstein, p. 373, 1992).

Then, the optimal hedging ratio is given by:

Proposition 8 *The optimal futures position strategy is θ^{TWSDU}, where*

$$\begin{aligned}\theta_t^{TWSDU} &= -\frac{(J_{ww}+J_{wx})+k(J)\left(J_w^2+J_wJ_x\right)}{(J_{ww}+2J_{wx}+J_{xx})+k(J)\left(J_w^2+2J_wJ_x+J_x^2\right)}\times \\ &\quad (\upsilon_t\upsilon_t')^{-1}\left[\upsilon_t\sigma_t'\pi_t+\frac{J_w+J_x}{(J_{ww}+J_{wx})+k(J)\left(J_w^2+J_wJ_x\right)}m_t\right].\end{aligned}$$

Proof. *In this case the relevant HJB*[22] *is*

$$\mathcal{A}_dJ+\frac{1}{2}k(J)\left(J_{z_t}'\Sigma J_{z_t}\right)=0.$$

Then the proof follows exactly the same lines to derive equation 7. ■

This expression is similar to the formula found in Section 3. In addition, because this is a concave transformation, the lemmas proved in Section 4 must hold. We just point out the following:

[22] *See appendix C for a proof.*

Lemma 1 *Given the assumptions about utility and wealth in this section,*

$$\begin{aligned} 0 &> \frac{J_w}{J_{ww}+k(J)J_w^2} = \\ &= \frac{E_t\left[h'\left(U\left(W_{T-t}^{\theta t}\right)\right)U'\left(W_{T-t}^{\theta t}\right)\right]}{E_t\left[h''\left(U\left(W_{T-t}^{\theta t}\right)\right)\left[U'\left(W_{T-t}^{\theta t}\right)\right]^2+h'\left(U\left(W_{T-t}^{\theta t}\right)\right)U''\left(W_{T-t}^{\theta t}\right)\right]}. \end{aligned}$$

Proof. *See appendix D.* ■

Consequently the hedge ratio simplifies enormously.

Proposition 9 *Given our assumption on the budget constraint and utility, the optimal hedging ratio is given by*

$$\theta_t^{TWSDU} = -\exp\left[-r\left(T-t\right)\right]\left(\upsilon_t\upsilon_t'\right)^{-1}\left[\upsilon_t\sigma_t'\pi+\frac{J_w}{J_{ww}+k(J)J_w^2}m_t\right]. \quad (16)$$

Proof. *Apply the corollaries of Section 4.* ■

This formula shows that the pure hedging demand is not affected by the SDU, similarly to the case of terminal wealth. Of course, if $k(J)=0$, we return to the standard case. $J_{ww}+k(J)J_w^2<0$ is as expected. But since we do not know the sign of J_{ww}, it is not obvious that the risk aversion increases. Moreover, if $m_t=0$, then there is no effect of SDU on the optimal hedge ratio and the analysis of DJ that the demand for futures is based uniquely on the hedge they provide holds.

5.1 EXPONENTIAL RISK-ADJUSTMENT: CRRA AND CARA

In this section, we give two simple examples for concreteness, where we specify $h(u) = -e^{-\rho u}$, with $\rho > 0$. In both, we show that the risk aversion increases, causing a decreasing on the pure speculative demand.

First, suppose that $u(w) = -e^{-\gamma w}$, where $\gamma > 0$. Then[23]

$$\begin{aligned} R &= -\left(\frac{J_{ww}}{J_w} + k(J)J_w\right) = \\ &= \gamma\left(1 + \rho\frac{E_t\left[\exp\left(-\rho U - 2\gamma W^{\theta t}_{T-t}\right)\right]}{E_t\left[\exp\left(-\rho U - \gamma W^{\theta t}_{T-t}\right)\right]}\right) \end{aligned}$$

We may interpret R as the global risk aversion parameter, as we have done in Section 3. From this expression, we can see clearly that, with certainty equivalent, we will need numerical methods to find the optimal hedge ratio. Also, we may realize that the risk aversion parameter increases by the proportion $\rho\frac{E_t\left[\exp\left(-\rho U - 2\gamma W^{\theta t}_{T-t}\right)\right]}{E_t\left[\exp\left(-\rho U - \gamma W^{\theta t}_{T-t}\right)\right]} > 0$. Consequently the pure speculative demand decreases[24]. Moreover, if $\rho = 0$, we are back to the Terminal Value-type utility example, given in the last section.

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[23] We suppress the TWSDU superscript of θ for simplicity.

[24] We are not considering equilibrium effects.

Second, suppose $u(w) = \frac{w^{1-\gamma}}{1-\gamma}, \gamma > 0$. Then

$$\begin{aligned} R &= -\left(\frac{J_{ww}}{J_w} + k(J)J_w\right) = \\ &= \gamma \frac{E_t\left[e^{-\rho U}\left(W_{T-t}^{\theta t}\right)^{-1-\gamma}\right]}{E_t\left[e^{-\rho U}\left(W_{T-t}^{\theta t}\right)^{-\gamma}\right]} \left(1 + \frac{\rho}{\gamma} \frac{E_t\left[e^{-\rho U}\left(W_{T-t}^{\theta t}\right)^{-2\gamma}\right]}{E_t\left[e^{-\rho U}\left(W_{T-t}^{\theta t}\right)^{-1-\gamma}\right]}\right). \end{aligned}$$

The risk aversion increases again, because $\frac{E_t\left[e^{-\rho U}\left(W_{T-t}^{\theta t}\right)^{-1-\gamma}\right]}{E_t\left[e^{-\rho U}\left(W_{T-t}^{\theta t}\right)^{-\gamma}\right]} > \frac{E_t\left[\left(W_{T-t}^{\theta t}\right)^{-1-\gamma}\right]}{E_t\left[\left(W_{T-t}^{\theta t}\right)^{-\gamma}\right]}$ (see appendix E for a proof), and also because of the additional term between parentheses. In particular, with log-utility, $\gamma = 1$, the risk aversion coefficient increases by a proportion somewhat greater than ρ.

5.2 EQUILIBRIUM

Just to complete our analysis, we consider an economy with a finite number of agents, I. Then, we obtain:

$$\theta_{it}^{TWSDU} = -e^{-r(T_i-t)}\left(\upsilon_t\upsilon_t'\right)^{-1}\left[\upsilon_t\sigma_t'\pi_{it} - R_{it}^{-1}m_t\right],$$

where $-R_{it}^{-1} = \frac{J_w^i}{J_{ww}^i + k^i(J^i)J_w^{i2}}$.

Market clearing, $\sum_{i=1}^{I}\theta_{it} = 0$, implies that:

$$m_t = \upsilon_t\sigma_t'\frac{\sum_{i=1}^{I}\omega_{it}\pi_{it}}{\sum_{i=1}^{I}\omega_{it}R_{it}^{-1}}. \quad (17)$$

The same analysis of Section 4 applies, and we do not repeat here.

Of course, replacing equation 17 into equation 7 gives:

$$\theta_{jt}^{TWSDU} = -e^{-r(T_i-t)} \left(\upsilon_t \upsilon_t'\right)^{-1} \upsilon_t \sigma_t' \left[\pi_{jt} - R_{jt}^{-1} \frac{\sum_{i=1}^{I} \omega_{it} \pi_{it}}{\sum_{i=1}^{I} \omega_{it} R_{it}^{-1}} \right].$$

6 CONCLUSIONS

In this paper we study the dynamic hedging problem using three different utility specifications: stochastic differential utility, terminal wealth utility, and a proposed utility which links both approaches. In all cases, we assume Markovian prices, as in Adler and Determple (1988). As a consequence of this assumption, we escape from a myopic hedging problem at each time. Furthermore, depending on the specification of the utility function, we must use different Hamilton-Jacobi-Bellman, HJB, equations.

Stochastic differential utility, SDU, where we maximize consumption over time as in Ho (1984), impacts the pure hedging demand ambiguously, because SDU parameters add both in the denominator and the numerator of the optimal ratio. We see that SDU decreases the pure speculative demand, because risk aversion increases. We also show that consumption decision is independent of the hedging decision in the sense that we can split the program into two independent programs, one for the consumption and the other for the optimal hedging. In this case, if the drift of futures prices is zero, there is no obvious impact on the optimal hedge.

In the second-type utility case, we derive a general and compact hedging formula, which nests all cases studied in Duffie and Jackson (1990). This formula may include the following particular assumptions found in DJ: Gaussian prices, mean-variance utility, and log-normal prices. We specialize to obtain their formulas.

With the appropriately modified Terminal Utility, we find a compact formula for hedging, which makes the second-type utility approach a special case, and show that the pure hedging demand is not impacted by this specification. We also see that, with CRRA- and CARA-type utilities, the risk aversion increases and, consequently the pure speculative demand decreases. If futures price are martingales, then such modification plays no role in determining the hedging allocation, giving exactly the same strategy of the Terminal Value environment.

In the appendix, we derive the relevant Bellman equation for each case, using semigroup techniques.

Although this is mainly a theoretical work, we might suggest some empirical applications of our model. For instance, we can try to simulate the hedge ratio and compare with some benchmark, varying the parameters of our model, provided some stationary assumptions are satisfied. Another example is if we can compare our model with alternative ones and try to infer its efficiency. In both cases, we could suggest the optimal hedging strategy to be followed by the hedger, and perhaps to say something about its possible results. Indeed, we believe that hedgers might be interested in testing the efficiency of this structure.

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Appendix A - Infinitesimal Generators

Here we present some technical concepts that we have used along the paper. First we present the infinitesimal generator approach, following closely Ait-Sahalia, Hansen and Scheinkman (2002). The HJB equation is easier to derive using this approach. Then we present the stochastic differential utility, according to the authors already cited.

Let $(\Omega, \Im, P)$ denote a probability space, Ξ a locally compact metric space with a countable basis E, a σ-field of Borelians in Ξ, I an interval of the real line, and for each $t \in I$, X_t is a stochastic process such that $X_t : (\Omega, \Im, P) \to (\Xi, E)$ is a measurable function, where (Ξ, E) is the state space.

Definition 2 *$Q : (\Xi, E) \to [0, \infty]$ is a transition probability if $Q(x, \cdot)$ is a probability measure in Ξ, and $Q(\cdot, B)$ is measurable, for each $(x, B) \in (\Xi \times E)$.*

Definition 3 *A transition function is a family $Q_{s,t}$, $(s,t) \in I^2, s < t$ that satisfies for each $s < t < u$ the Chapman-Kolmogorov equation:*

$$Q_{s,u}(x, B) = \int Q_{t,u}(y, B) \, Q_{s,t}(x, dy) \, .$$

A transition function is homogeneous if $Q_{s,t} = Q_{s',t'}$ whenever $t - s = t' - s'$.

Definition 4 *Let $\Im_t \in \Im$ be an increasing family of σ-algebras, and X a stochastic process that is adapted to $\Im_t$. X is Markov with transition function $Q_{s,t}$ if for each non-negative Borel measurable $\phi : \Xi \to \mathbb{R}$ and each $(s,t) \in I^2, s < t$*

$$E[\phi(X_t) \,|\Im_s] = \int \phi(y) \, Q_{s,t}(X_s, dy) \, .$$

Assume that Q_t is a homogeneous transition function and L is a vector space of real valued functions such that for each test function, $\phi \in L$, $\int \phi(y) \, Q_t(x, dy) \in L$. Now, for each t define the conditional expectation operator

$$\mathcal{T}_t\phi(x) = E[\phi(y_t)|x_0 = x] = \int \phi(y) Q_t(x, dy).$$

The Chapman-Kolmogorov equation guarantees that the linear operators $\mathcal{T}_t$ satisfy $\mathcal{T}_{t+s} = \mathcal{T}_t\mathcal{T}_t$.

With this in hand we can propose a parameterization for Markov processes. To do this, let $(L, \|\cdot\|)$[25] be a Banach space.

Definition 5 *A one parameter family of linear operators in L,* $\{\mathcal{T}_t : t \geq 0\}$ *is called strongly continuous contraction semigroup if*

a. $\mathcal{T}_0 = I$;

b. $\mathcal{T}_{t+s} = \mathcal{T}_t\mathcal{T}_s$ *for all* $s, t \geq 0$*;*

c. $\lim_{t\to 0}\mathcal{T}_t\phi = \phi$*; and*

d. $\|\mathcal{T}_t\| \leq 1$.

In general, the semigroup of conditional expectations determine the finite-dimensional distributions of the Markov process, as we can infer from Ethier and Kurtz (1986, Proposition 1.6 of chapter 4).

Now we can define infinitesimal generators. A generator describes the instantaneous evolution of a semigroup. Heuristically we could think of it as the derivative of the operator $\mathcal{T}$ with respect to time, when it goes to zero.

Definition 6 *The infinitesimal generator of a semigroup* $\mathcal{T}_t$ *on a Banach*

[25] Notice that V is contained in this space.

space L is the (possibly unbounded) linear operator $\mathcal{A}$ defined by:

$$\mathcal{A}\phi = \lim_{t \downarrow 0} \frac{\mathcal{T}_t \phi - \phi}{t}.$$

The domain $D(\mathcal{A})$ is the subspace of L for which this limit exists.

If $\mathcal{T}_t$ is a strongly continuous contraction semigroup, we can reconstruct $\mathcal{T}_t$ using its infinitesimal generator $\mathcal{A}$ (see Ethier and Kurtz (1986), Proposition 2.7 of Chapter 1). Thus the Markov process can be parameterized using $\mathcal{A}$.

The space $\mathcal{C}$ of continuous functions on a compact state space endowed with the sup-norm is a common domain for a semigroup. For instance, the generator $\mathcal{A}_d$ of a multivariate diffusion process is an extension of the second-order differential operator:

$$\mathcal{A}_d \phi(x) \equiv \frac{d}{dt} E_x \left[\phi(X_t)\right]_{|t=0+} = \mu \cdot \phi_x + \frac{1}{2} tr \left[\Sigma \phi_{xx}\right],$$

where

$$dx_t = \mu(x_t)\, dt + \Lambda(x_t)\, dB_t,$$

tr is the trace operator, and

$$\Lambda(x_t)\, \Lambda'(x_t) = \Sigma(x_t).$$

For a formal proof see, for instance, Oksendal (1995). For more details, see Ait-Sahalia, Hansen and Scheinkman (2002), and Hansen and Scheinkman

(2002).

The domain of this second order differential operator is restricted to the space of twice continuously differentiable functions with compact support.

Appendix B - Stochastic Differential Utility

The stochastic differential utility formulation by Duffie and Epstein (1992) is a continuous-time analogue of the utility model that appears in Epstein and Zin (1989). As it is known, the main advantage of this framework compared to the standard one is that we can disentangle risk aversion from intertemporal substitutability. We now present the main characteristics of this approach, based on Duffie and Epstein (1992).

Define $J(z_t)$ as the value function, where z is the state variable. The standard additive utility specification in which the utility at time t for a consumption process c is defined by

$$J(z_t) = E_t \left[\int_{s \geq t} e^{-\delta(s-t)} u\left(c_s\right) ds \right], t \geq 0,$$

where E_t denotes expectation given information available at time t, and δ is the discount rate.

The more general utility functions are named *recursive*, exhibit intertemporal consistency and admit the Hamilton-Jacobi-Bellman's characterization of optimality. The stochastic differential utility $U : V \rightarrow \mathbb{R}$ is defined as follows by two primitive functions: $f : \mathbb{R}^D \times \mathbb{R} \rightarrow \mathbb{R}$ and $k : \mathbb{R} \rightarrow \mathbb{R}$, as already stated. When well defined, the utility process, J, for a given consumption process, c, is the unique Ito process J having a stochastic differential representation of the form

$$dJ(z_t) = \left[-f(c_t, J(z_t)) - \frac{1}{2}k(J)J'_{z_t}\Sigma J_{z_t}\right]dt + J'_{z_t}\Lambda\left(z_t\right)dB_t,$$

where the subscript of J indicates derivative with respect to the argument.

In this framework the pair (f, k) is called "aggregator", that determines the consumption process, c, such that the utility process, J, is the unique solution to

$$J(z_t) = E_t\left[\int_{s\geq t}\left\{f\left[c_s, J\left(z_s\right)\right] + \frac{1}{2}k(J)J'_{z_s}\Sigma J_{z_s}\right\}ds\right], t \geq 0.$$

We think of $J(z_t)$ as the continuation utility of c at time t, conditional on current information; and $k(J)$ as the variance multiplier, applying a penalty (or reward) as a multiple of the utility "volatility" $J'_z\Sigma J_z$. In a discrete time setting, we could say that at time t, the intertemporal utility $J\left(\cdot, t+1\right)$ for the period ahead and beyond is a random variable. Thus first the agent computes the certainty equivalent, $m\left(\sim J\left(\cdot, t+1\right)|\Im_t\right)$, of the conditional distribution $\sim J\left(\cdot, t+1\right)|\Im_t$ of $J\left(\cdot, t+1\right)$, given information $\Im_t$ at time t. Then (s)he combines the latter with c_t via the aggregator. The function f encodes the intertemporal substitutability of consumption and other aspects of "certainty preferences", also generating a collateral risk attitude under uncertainty. The certainty equivalent function, m, encodes the risk aversion in the sense described in Epstein and Zin (1989).

In this paper, our expected-utility based specification is:

$$m(\sim J) = h^{-1}\left(E_t\left[h(J)\right]\right),$$

where J is a real-valued integrable random variable, $\sim J$ denotes its distribution, and h has been already defined in Section 2.

It easy to see that, if $f(c, j) = u(c) - \delta j$, with $k(J) = 0$, then we are back to the standard formulation.

Appendix C - Hamilton-Jabobi-Bellman Equation

In this subsection we derive the relevant HJB equations for our problems

First, we state the Hille-Yosida Theorem:

Theorem 1 *Let $\mathcal{T}_t$ be a strongly continuous contraction semigroup on L with generator $\mathcal{A}$. Then*

$$(\delta\mathcal{I} - \mathcal{A})^{-1} g = \int_0^{\infty} e^{-\delta t}\mathcal{T}_s g ds,$$

for all $g \in L$ and $\delta > 0$.

Proof. *See Ethier and Kurtz (1986), Proposition 2.1 of chapter 1.* ■

Standard Additive Utility

If we apply this theorem to a standard additive utility function, say, $J(z_t) = E_t\left[\int_{s\geq t} e^{-\delta(s-t)} u(c_s)\, ds\right], t \geq 0$, we obtain the following HJB (standard) equation:

Proposition 10 *Assume $u \in L$ and $\delta > 0$. Then the HJB equation is:*

$$u(c_t) - \delta J + \mathcal{A}_d J = 0,$$

Proof. *(You may apply the theorem as well) Define the value function as:*

$$J(z_t) = E_t \left[\int_{s \geq t} e^{-\delta(s-t)} u\left(c_s\right) ds \right], t \geq 0.$$

Consequently:

$$J(z_{t+\varepsilon}) = E_{t+\varepsilon} \left[\int_{s \geq t+\varepsilon} e^{-\delta(s-t-\varepsilon)} u\left(c_s\right) ds \right], t+\varepsilon \geq 0.$$

Take the conditional expectations at t:

$$\mathcal{T}_\varepsilon J(z_t) = E_t \left[\int_{s \geq t+\varepsilon} e^{-\delta(s-t-\varepsilon)} u\left(c_s\right) ds \right].$$

Subtract the first equation from the last one:

$$\mathcal{T}_\varepsilon J\left(z_t\right) - J\left(z_t\right) = E_t \left[\int_{s \geq t+\varepsilon} e^{-\delta(s-t-\varepsilon)} u\left(c_s\right) ds - \int_{s \geq t} e^{-\delta(s-t)} u\left(c_s\right) ds \right].$$

Divide by ε and take the limit when $\varepsilon \downarrow 0$:

$$\lim_{\varepsilon \downarrow 0} \frac{LHS}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{\mathcal{T}_\varepsilon J\left(z_t\right) - J\left(z_t\right)}{\varepsilon} = \mathcal{A}_d J.$$

Now, let us analyze the RHS:

$$\begin{aligned} RHS &= E_t\left[\int_{s\geq t+\varepsilon} e^{-\delta(s-t-\varepsilon)}u\left(c_s\right)ds\right] - E_t\left[\int_{s\geq t} e^{-\delta(s-t)}u\left(c_s\right)ds\right] = \\ &= E_t\left[\int_{s\geq t+\varepsilon} e^{-\delta(s-t)}\left(e^{\delta\varepsilon}u\left(c_s\right) - u\left(c_s\right)\right)ds - \int_t^{t+\varepsilon} e^{-\delta(s-t)}u\left(c_s\right)ds\right]. \end{aligned}$$

Divide by ε and take the limit when $\varepsilon \downarrow 0$:

$$\begin{aligned} \lim_{\varepsilon\downarrow 0}\frac{RHS}{\varepsilon} &= \lim_{\varepsilon\downarrow 0}\frac{E_t\left[\int_{s\geq t+\varepsilon} e^{-\delta(s-t)}\left(e^{\delta\varepsilon}u\left(c_s\right) - u\left(c_s\right)\right)ds - \int_t^{t+\varepsilon} e^{-\delta(s-t)}u\left(c_s\right)ds\right]}{\varepsilon} = \\ &= \lim_{\varepsilon\downarrow 0} E_t\left[\delta\int_{s\geq t+\varepsilon} e^{-\delta(s-t)}e^{\delta\varepsilon}u\left(c_s\right)ds - \left(u\left(c_{t+\varepsilon}\right) - e^{-\delta\varepsilon}u\left(c_{t+\varepsilon}\right)\right)\right] - u\left(c_t\right) = \\ &= \delta J\left(z_t\right) - u\left(c_t\right). \end{aligned}$$

In the second line, we have used the L'Hôpital's rule.

Hence the result follows. ■

Standard Recursive Utility

The same proof may be applied with recursive utility,

$$E_t\int_{s\geq t}\left[f\left(c_s, J\left(z_s\right)\right) + \frac{1}{2}k\left(J\left(z_s\right)\right)J'_{z_s}\Sigma J_{z_s}\right]ds.$$

Proposition 11 *Assume $f, k \in L$ and $\delta > 0$. Then the HJB equation is:*

$$u(c_t) - \delta J + \mathcal{A}_d J + \frac{1}{2}k(J)J'_{z_t}\Sigma J_{z_t} = 0,$$

Proof. *The idea is the same as before. We just have to show what* *happens to:*

$$E_t \int_{s \geq t+\varepsilon} \frac{1}{2} k(J) J'_{z_s} \Sigma J_{z_s} ds - E_t \int_{s \geq t} \frac{1}{2} k(J) J'_{z_s} \Sigma J_{z_s} ds =$$

$$= -E_t \int_t^{t+\varepsilon} \frac{1}{2} k(J) J'_{z_s} \Sigma J_{z_s} ds$$

Divide by ε *and take the limit when* $\varepsilon \downarrow 0$*:*

$$-\lim_{\varepsilon \downarrow 0} \frac{E_t \int_t^{t+\varepsilon} \frac{1}{2} k(J) J'_{z_s} \Sigma J_{z_s} ds}{\varepsilon} = -\frac{1}{2} k(J) J'_{z_t} \Sigma J_{z_t}$$

■

For another derivation attack, we refer to Duffie and Epstein (1992). For a heuristic proof:

$$J(z_t) = \varepsilon u(c_t) + e^{-\delta \epsilon} h^{-1} (\mathcal{T}_\epsilon h(J)).$$

Consequently:

$$\begin{aligned} 0 &= \lim_{\varepsilon \downarrow 0} \left[u(c_t) + \frac{e^{-\delta \epsilon} h^{-1} (\mathcal{T}_\epsilon h(J)) - J}{\epsilon} \right] = \\ &= \lim_{\varepsilon \downarrow 0} \left[u(c_t) + \frac{-\delta e^{-\delta \epsilon} h^{-1} [\mathcal{T}_\epsilon h(J)] + e^{-\delta \epsilon} \frac{\mathcal{A}_d h(J)}{h'(J)}}{1} \right] = \\ &= u(c_t) - \delta J(z_t) + \frac{\mathcal{A}_d h(J)}{h'(J)}. \end{aligned}$$

The proof follows now by the same lines as in Proposition 14.

Degenerated Additive Utility

We apply the same kind of proof as before.

Proposition 12 *The relevant HJB equation, which maximizes* $J(z_t) = E_t\left[U\left(W_T^{\theta t}\right)\right]$ *is given by:*

$$\mathcal{A}_d J = 0.$$

Proof. *This is trivial. Just notice that* $J(z_t) = E_t\left[U\left(W_T^{\theta t}\right)\right]$, *implying*

$$J(z_{t+\varepsilon}) = E_{t+\varepsilon}\left[U\left(W_T^{\theta t+\varepsilon}\right)\right].$$

Thus

$$\mathcal{T}_\varepsilon J(z_t) - J(z_t) = 0.$$

And the proof follows the LHS of the second Proposition in this section.

■

The reader is referred to Krylov (1980, chapter 5) for another derivation strategy.

Modified Degenerated Aditive Utility

Here we can apply either of two equivalent specifications.

Proposition 13 *Suppose we want to define the following value function*

$$J(z_t) = E_t\left[U\left(W_T^{\theta t}\right)\right] + \frac{1}{2}E_t \int_{s \geq t}^{T} k\left(J\left(z_s\right)\right) J'_{z_s} \Sigma J_{z_s} ds,$$

with boundary condition $J(z_T) = U(w)$.

Then, the relevant HJB equation is

$$\mathcal{A}_d J + \frac{1}{2}k(J)J'_{z_t}\Sigma J_{z_t} = 0.$$

Proof. *See earlier proofs.* ■

We could alternatively specify the following:

Proposition 14 *Suppose we want to define the following value function*

$$J(z_t) = h^{-1}\left(E_t\left[h\left(U\left(W_T^{\theta t}\right)\right)\right]\right),$$

$J(z_T) = U(w)$.

Then the relevant HJB equation is:

$$\mathcal{A}_d J + \frac{1}{2}k(J)J'_{z_t}\Sigma J_{z_t} = 0.$$

Proof. *Observe that:*

$$h\left(J\left(z_t\right)\right) = E_t\left[h\left(U\left(W_T^{\theta t}\right)\right)\right]$$

Now we follow the same procedure as before.

$$h\left(J\left(z_{t+\varepsilon}\right)\right)=E_{t+\varepsilon}\left[h\left(U\left(W_T^{\theta t+\varepsilon}\right)\right)\right].$$

Take the conditional expectations at t, and subtract the first equation:

$$\mathcal{T}_\varepsilon h\left(J(z_t)\right)-h\left(J\left(z_t\right)\right)=0.$$

Thus:

$$\lim_{\varepsilon\downarrow 0}\frac{LHS}{\varepsilon}=\lim_{\varepsilon\downarrow 0}\frac{\mathcal{T}_\varepsilon h\left(J(z_t)\right)-h\left(J\left(z_t\right)\right)}{\varepsilon}=\mathcal{A}_d h\left(J(z_t)\right)=0$$

Since $\frac{\partial h(J)}{\partial z}=h'(J)J_z$ *and* $\frac{\partial^2 h(J)}{\partial z\partial z'}=h''(J)J_zJ_z'+h'(J)J_{zz}$, *then*

$$\begin{aligned}0 &= \frac{\mathcal{A}_d h\left(J\right)}{h'(J)}=\frac{1}{h'(J)}\left[\mu\cdot\frac{\partial h}{\partial z_t}+\frac{1}{2}tr\left(\Sigma\frac{\partial^2 h\left(J\right)}{\partial z_t\partial z_t'}\right)\right]=\\ &= \mu\cdot J_{z_t}+\frac{1}{2}tr\left(\Sigma J_{z_tz_t}\right)+\frac{h''(J)}{2h'(J)}\left[J_{z_t}'\Sigma J_{z_t}\right]=\\ &= \mathcal{A}_d J+\frac{1}{2}k(J)J_{z_t}'\Sigma J_{z_t},\end{aligned}$$

Then it follows that:

$$0=\frac{\mathcal{A}_d h\left(J\right)}{h'\left(J\right)}=\mathcal{A}_d J\left(z_t\right)+\frac{1}{2}k(J)J_{z_t}'\Sigma J_{z_t}.$$

■

Appendix D - Proof of Lemma 2

Proof. Since $J(z_t) = h^{-1}\left(E_t\left[h\left(U\left(W_T^{\theta t}\right)\right)\right]\right)$, then $h(J(z_t)) = E\left[h\left(U\left(W_{T-t}^{\theta t}\right)\right)\right]$. Take the derivatives in both sides[26]:

$$h'(J)J_w = E\left[h^{'}\left(U\left(W_{T-t}^{\theta t}\right)\right)U^{'}\left(W_{T-t}^{\theta t}\right)\frac{\partial W_{T-t}^{\theta t}}{\partial w}\right];$$

$$h'(J)J_x = E\left[h^{'}\left(U\left(W_{T-t}^{\theta t}\right)\right)U^{'}\left(W_{T-t}^{\theta t}\right)\frac{\partial W_{T-t}^{\theta t}}{\partial x}\right].$$

We provide only the second derivative for J_{wx}:

$$\begin{aligned} h''(J)\,J_w J_x + h'(J)\,J_{wx} &= E\Bigg[\Bigg(h''\left(U\left(W_{T-t}^{\theta t}\right)\right)\left[U^{'}\left(W_{T-t}^{\theta t}\right)\right]^2 + \\ &\quad h'\left(U\left(W_{T-t}^{\theta t}\right)\right)U''\left(W_{T-t}^{\theta t}\right)\Bigg)\frac{\partial W_{T-t}^{\theta t}}{\partial w}\frac{\partial W_{T-t}^{\theta t}}{\partial x} + \\ &\quad h'\left(U\left(W_{T-t}^{\theta t}\right)\right)U'\left(W_{T-t}^{\theta t}\right)\frac{\partial^2 W_{T-t}^{\theta t}}{\partial w \partial x}\Bigg]. \end{aligned}$$

If we apply Proposition 3, this simplifies to

$$0 < J_w = \frac{E\left[h^{'}\left(U\left(W_{T-t}^{\theta t}\right)\right)U^{'}\left(W_{T-t}^{\theta t}\right)\right]}{h'(J)};$$

[26] Again we claim that the conditions for differentiability inside the expectations are satisfied because of our assumptions on h. See DJ (1990, appendix).

$$\begin{aligned} 0 &< J_x = \left(e^{r(T-t)} - 1\right) \frac{E\left[h'\left(U\left(W_{T-t}^{\theta t}\right)\right) U'\left(W_{T-t}^{\theta t}\right)\right]}{h'(J)} = \\ &= \left(e^{r(T-t)} - 1\right) J_w; \end{aligned}$$

$$\begin{aligned} 0 &> h''(J) J_w^2 + h'(J) J_{ww} = \\ &= E\left[h''\left(U\left(W_{T-t}^{\theta t}\right)\right)\left[U'\left(W_{T-t}^{\theta t}\right)\right]^2 + h'\left(U\left(W_{T-t}^{\theta t}\right)\right) U''\left(W_{T-t}^{\theta t}\right)\right] \Longrightarrow \\ &\Longrightarrow J_{ww} + k(J) J_w^2 = \frac{E\left[h''\left(U\left(W_{T-t}^{\theta t}\right)\right)\left[U'\left(W_{T-t}^{\theta t}\right)\right]^2 + h'\left(U\left(W_{T-t}^{\theta t}\right)\right) U''\left(W_{T-t}^{\theta t}\right)\right]}{h'(J)}; \end{aligned}$$

$$\begin{aligned} 0 &> h''(J) J_x^2 + h'(J) J_{xx} = \left(e^{r(T-t)} - 1\right)^2 \times \\ &= E\left[h''\left(U\left(W_{T-t}^{\theta t}\right)\right)\left[U'\left(W_{T-t}^{\theta t}\right)\right]^2 + h'\left(U\left(W_{T-t}^{\theta t}\right)\right) U''\left(W_{T-t}^{\theta t}\right)\right] \Longrightarrow \\ &\Longrightarrow k(J) J_w^2 + \frac{J_{xx}}{\left(e^{r(T-t)} - 1\right)^2} = J_{ww} + k(J) J_w^2 \Longrightarrow \\ &\Longrightarrow J_{xx} = \left(e^{r(T-t)} - 1\right)^2 J_{ww}; \end{aligned}$$

$$
\begin{aligned}
0 &> h''(J)\, J_w J_x + h'(J)\, J_{wx} = \left(e^{r(T-t)} - 1\right) \times \\
&= E\left[h''\left(U\left(W^{\theta t}_{T-t}\right)\right)\left[U'\left(W^{\theta t}_{T-t}\right)\right]^2 + h'\left(U\left(W^{\theta t}_{T-t}\right)\right)U''\left(W^{\theta t}_{T-t}\right)\right] \Longrightarrow \\
&\Longrightarrow k(J)\, J_w^2 + \frac{h'(J)\, J_{wx}}{\left(e^{r(T-t)} - 1\right)} = J_{ww} + k(J) J_w^2 \Longrightarrow \\
&\Longrightarrow J_{xw} = \left(e^{r(T-t)} - 1\right) J_{ww}.
\end{aligned}
$$

■

Appendix E - Risk Aversion Increases with Power Utility

Proposition 15 *With Power utility and exponential risk adjustment, risk aversion increases.*

Proof. *Clearly* $cov(e^{-\rho U}, \left(W^{\theta t}_{T-t}\right)^{-a-\gamma}), a = 0, 1,$ *is positive, since* $\frac{d}{dW}\exp\left[-\rho\frac{W^{1-\gamma}}{1-\gamma}\right] < 0$*, and* $\frac{d}{dW}W^{-a-\gamma} < 0$*. Now, to simplify the notation, let* $q \equiv e^{-\rho U}$*,* $p \equiv \left(W^{\theta t}_{T-t}\right)^{-1-\gamma}$*, and* $g \equiv \left(W^{\theta t}_{T-t}\right)^{-\gamma}$*. We want to prove that*

$$
\frac{E(pq)}{E(gq)} \geq \frac{E(p)}{E(g)}.
$$

Since

$$
Cov(p, q) \geq 0 \Longrightarrow E(pq) \geq E(p)\, E(q).
$$

And the same holds for $E(gq)$. *Since each expectation is positive:*

$$\begin{aligned} E(g)\,E(pq) &\geq E(p)\,E(q)\,E(g)\,; \\ E(p)\,E(gq) &\geq E(p)\,E(q)\,E(g)\,. \end{aligned}$$

Subtract one from the other and the assertion is proved. ■