Infinite Horizon Incomplete Markets: Long-Lived Assets, Default and Bubbles.

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July, 2002

PRELIMINARY and INCOMPLETE

*A.Araujo and J.P.Torres-Martínez acknowledge financial support from the CNPq, Brazil. M.R.Páscoa bene-

1 Introduction

Araujo, Páscoa and Torres-Martínez (2000) have previously established that in infinite horizon economies with short-lived assets backed by physical collateral equilibrium always exists. Ponzi shemes are avoided by the presence of physical collateral and, therefore, there is no need to impose debt restrictions or a priori transversality conditions. We intend now to address the case of infinite-lived assets and extend the model to allow also for financial collateral and default penalties. Various interesting questions arise in this new context. Are collateral requirements still sufficient to prevent agents from rolling on their debts? Can debts be renegociated when default occurs? Are assets in positive net supply more prone now to price bubbles, than in the default-free case studied by Santos and Woodford (1997) or Magill and Quinzii(1996)? Can bubbles propagate easier across a collateralized economy?

Sequential economies with infinite-lived assets have been studied for quite a long time in finance and macroeconomics. The overlapping generations model by Samuelson (1958) and Gale (1973), as well as the infinite-lived consumers model by Bewley (1980), are two benchmarks in modern macroeconomics. In these models money has a positive value in equilibrium, although its fundamental value (the discounted stream of future returs) is zero. This excess of the price of an asset over its fundamental value is called a bubble and has been observed also in econometric studies of perpetuities, in spite of the fact that prices always satisfy a discounted dividends relation with respect to the market values (accrued of returns) of the period that immediately follows.

Only very recently were bubbles analised in the framework of the modern general equilibrium theory of incomplete markets. According to Santos and Woodford (1997) and Magill and Quinzii (1996), assets in zero net supply are subject to bubbles, whereas assets in positive net supply (such as money in the above mentioned models) are free of bubbles under a reasonable assumption. These results question the scope of the conclusions obtained by Samuelson (1958), Gale (1973) and Bewley (1980). The crucial assumption behind the new results was that financial markets should be sufficiently productive so that aggregate endowment could be super-replicated by a portfolio plan with finite cost. This hypothesis resumed, in the complete markets case, to aggregate endowment having a finite present value, under the unique non-arbitrage discount factor process, and this did not happen in the above models by Samuelson, Gale or Bewley.

In incomplete markets, discount processes are not uniquely determined. A finite infimum for the costs of portfolios that super-replicate aggregate endowment is a finite upper bound for the fundamental value of aggregate endowment. Collateralized short-lived assets economies have

fited from a Nova Forum grant and support from project POCTI/36525 (FCT/FEDER).

equilibria even when there is no finite lower bound for the fundamental value of aggregate endowment. In fact, debt or a priori transversality constraints ensure this finite lower bound, but were not required by Araujo, Páscoa and Torres-Martínez (2000). It is therefore interesting to check whether these constraints can also be dispensed with in the case of infinite-lived assets, leaving also more room for the impossibility of super-replicating aggregate endowment at finite cost.

Methodology and Summary of Results

The infinite horizon economy has countably many nodes, as in Araujo, Páscoa and Torres-Martínez (2000), but departs from this previous model when allowing for incomplete agents participation, financial collateral, tranching and utility penalties in case of default. To avoid cycles, where assets are mutually protected through financial collateral, we assume a pyramidal structure of financial collateral requirements having only physical collateral or default-free assets with bounded short-sales (node by node) in its ultimate protective layer.

When default occurs, assets subsequent returns have to be redefined. In fact, node by node, agents may pay their debts, surrender the collateral or choose a combination of these two solutions. In the absence of utility penalties, a combination whose be chosen only when the value of the debt is equal to the value of the collateral. In the presence of these penalties, combined outcomes are more common. In equilibrium, subsequent returns have to be redefined, or equivalently, it is as if a new asset appears which is now backed by the non-surrendered collateral and yields lower returns. That is, debts are renegotiated according to the mean combination of payments and surrender of collateral.

Our first results address existence of equilibrium. We assume that collateral requirements, in units of collateral per unit of protected asset, are endogenously adjusted along the life of the asset in order to meet a certain fixed margin requirement (loan/value of collateral). Moreover collateral requirements are bounded from below, node by node, and endowments uniformly bounded from above. When there are only durable goods beneath the whole pyramid of collateral, (i) equilibrium always exists in the absence of utility penalties, but, (ii) when these penalties are imposed, margin requirements have to be less than one and decrease whenever default occurs.

To see the intuition behind these results, notice that a borrower performs a joint operation - sell an asset and constitute collateral. In the absence of default penalties, the returns from this joint operation are nonnegative, since the future value of the collateral always exceeds what he decides to pay back (the minimum between the debt and this same future value of the collateral). Hence, the net price of the joint operation must be nonnegative in this case, that is, the current value of the collateral requirements can not be lower than the asset price. It is as if the borrower were buying the joint product loan - collateral. In this context, Ponzi schemes can be ruled out. However, when utility penalties are imposed, the returns from this joint operation are no longer nonnegative and, therefore, asset prices may exceed the current value of respective collateral. To avoid this possibility we assume that margin requirements are always less than one.

We can accommodate financial collateral at the ultimate layer of the pyramid, in the form of default-free securities, provided that we bound, uniformly, from below, endowments, and, from above, the number of assets traded at each node as well as returns and short-sales of default-free assets.

Our second set of results addresses the occurrence of bubbles. Prices of assets subject to default exhibit no bubbles, at a certain node, if there is a vector of deflators such that the discounted value at that node of collateral requirements at a future period T goes to zero as T goes to infinity. Due to the pyramidal collateral structure, it suffices to rule out bubbles in the underlying durable goods or default-free assets that are beneath the pyramid.

2 The Economy

The model of uncertainty and the characterization of commodities used here is essentially the one developed in Araujo, Páscoa, and Torres-Martínez (2000). Along the following lines we recall the main features of this model; introduce the possibility of incomplete household participation, as in Santos and Woodford (1996); and define the structure of multiperiod securities in the economy.

2.1 Uncertainty

We work in an economy with infinite horizon and discrete time. The set of trade periods is $T = \{0, 1, \ldots\}$. We suppose that there is no uncertainty at $t = 0$. There are S_1 states of nature at $t = 1$. In general, given a history of realization of the states of nature for the first $t-1$ periods: $\overline{s}_t = (s_0, s_1, \ldots, s_{t-1})$, there exist $S(\overline{s}_t)$ states of nature at period t. We suppose that these sets are *finite*.

An information set $\xi = (t, \overline{s}_t, s)$, where $t \in T$ and $s \in S(\overline{s}_t)$, is called a *node* of the economy. The period associated to node ξ is denoted by $\tilde{t}(\xi)$. The (unique) predecessor of node ξ is denoted by ξ^- and the only information set at $t = 0$ is ξ_0. Given a node $\xi = (t, \overline{s}_t, s)$, we denote by ξ^+ the set of immediate successors of ξ, that is, the set of nodes $\mu = (t+1, \overline{s}_{t+1}, s')$, where $\overline{s}_{t+1} = (\overline{s}_t, s)$. There exists a natural order in the information structure: given nodes $\xi = (t, \overline{s}_t, s)$

and $\mu = (t', \overline{s}_{t'}, s')$, we say that μ is a *successor* of the node ξ, and write $\mu \geq \xi$, if $t' \geq t$ and $\overline{s}_{t'} = (\overline{s}_t, \dots)$.

The set of nodes or information sets in the economy, called the *event-tree*, is denoted by $\mathcal{D}$. The set of nodes with period t is denoted by $\mathcal{D}_t$, and, given an information set ξ, $\mathcal{D}(\xi)$ denotes the sub-tree whose root is ξ (that is, the set of successors of the node ξ).

Finally, a set $\mathcal{D}' \subset \mathcal{D}$ will be called a *sub-tree* of the event-tree if there is a node $\xi \in \mathcal{D}'$, called the *root* of the sub-tree, such that $\mathcal{D}' \subset \mathcal{D}(\xi)$ and $\mathcal{D}' \cap \mathcal{D}(\mu) = \phi$, for all nodes $\mu \in \mathcal{D}(\xi) - \mathcal{D}'$.

2.2 Commodities

We suppose that at each node of the event-tree $\mathcal{D}$ the economy has a finite set $\mathcal{L}$ of commodities which can be consumed or stored. Besides, these commodities may suffer partial depreciation at the node branches; this depreciation varies according to whether the commodity is for consumption or storage. Note that if the depreciation factors for consumption and storage are the same, then the agents have no interest in storing commodities. Therefore, if one unit of the good $l \in \mathcal{L}$ is consumed at the node ξ, then at each node $\mu \in \xi^+$ we obtain an amount $(Y^c_\mu)_{l,l'}$ of the good l'. Analogously, $(Y^w_\xi)_{l,l'}$ denotes the amount of commodity l' that is obtained at the node ξ if one unit of the good l was stored at the node ξ^-. These structures are very general, allowing for instance for goods that are perishable, perfectly durable, simply one period older or perfectly storable but non-durable.

Spot markets for commodity negotiations are available at each node. We denote by $p_\xi = \big(p_{\xi,l} \ : \ l \in \mathcal{L}\big)$ the vector of spot prices at the node $\xi \in \mathcal{D}$.

2.3 Agents

Incomplete participation of the agents is allowed in the economy, and therefore at each node ξ of the event-tree there exists a *finite* set $\mathcal{H}(\xi)$ of agents able to trade in the spot markets.

We suppose that $\mathcal{H}(\xi)$ is a subset of the agents set $\mathcal{H}$, which is countable. We denote by $\mathcal{D}^h$ the sub-tree of nodes at which the agent $h \in \mathcal{H}$ can trade in spot markets. The root of the sub-tree $\mathcal{D}^h$ is denoted by ξ^h. We say that the agent h is *infinite-lived* if for all $t \in \mathbb{N}$ there exists a node $\xi \in \mathcal{D}^h$ such that $\tilde{t}(\xi) = t$. Otherwise we say that the agent h is *finite-lived*.

The set $\delta\mathcal{D}^h$ denotes the *terminal nodes* for the agent h. That is, the set of nodes $\xi \in \mathcal{D}^h$ for which $\mathcal{D}(\xi) \cap \mathcal{D}^h = \{\xi\}$ (if such nodes exist; otherwise we suppose that $\delta\mathcal{D}^h$ is empty).

Two technical restrictions that appear in the literature in Santos and Woodford (1997) are also needed:

a. For each agent $h \in \mathcal{H}$, if $\xi \in (\mathcal{D}^h - \delta\mathcal{D}^h)$ then $\xi^+ \subset \mathcal{D}^h$,

b. For each node $\xi \in \mathcal{D}$ there exists at least one agent $h \in \mathcal{H}$ for which $\xi \in (\mathcal{D}^h - \delta\mathcal{D}^h)$.

We denote by $\tilde{\mathcal{H}}(\xi)$ the set of agents able to negotiate assets at the node ξ. Hence, we have that $h \in \mathcal{H}(\xi)$ if and only if $\xi \in \mathcal{D}^h$, and $h \in \tilde{\mathcal{H}}(\xi)$ if and only if $\xi \in (\mathcal{D}^h - \delta\mathcal{D}^h)$.

At each node $\xi \in \mathcal{D}^h$, the agent h can choose a *collateral-free consumption allocation* $x^h(\xi) \in \mathbb{R}_+^{\mathcal{L}}$. We denote by $x^h = \left(x^h(\xi)\right)_{\xi\in\mathcal{D}^h}$ the collateral-free consumption plan of the agent h. At the nodes $\xi \in (\mathcal{D}^h - \delta\mathcal{D}^h)$ the agent h is also able to choose a *collateral-free storage allocation* $y^h(\xi) \in \mathbb{R}_+^{\mathcal{L}}$. A family of such allocations, $y^h = \left(y^h(\xi)\right)_{\xi\in(\mathcal{D}^h-\delta\mathcal{D}^h)}$, is called a collateral-free storage plan of the agent h.

We use the notation $X^h = \mathbb{R}_+^{\mathcal{D}^h\times\mathcal{L}}$ and $Y^h = \mathbb{R}_+^{(\mathcal{D}^h-\delta\mathcal{D}^h)\times\mathcal{L}}$ to denote respectively the consumption- and the storage-spaces of the agent h in the economy.

Each agent $h \in \mathcal{H}$ is characterized by an endowment process $w^h \in X^h$ and a utility function $U^h : X^h \to \mathbb{R}_+$ that represents his preferences in the consumption space X^h.

2.4 Assets

The set $\mathcal{J}$ of assets, including long-lived or infinite-lived real securities, that can be negotiated in the economy is the union of three disjoint sets:[1]

- The set $\mathcal{J}^b$ of assets that, not being subject to default, have exogenous bounds on short sales at each node where they are traded;

- The set $\mathcal{J}^d$ of assets which might suffer default but which are *individually* protected by collateral requirements. These collateral requirements are exogenously imposed on the agents; they may consist of consumption goods (Physical Collateral) as well as of assets (Financial Collateral);

- The set $\mathcal{J}^t$ of families of real short-lived assets (tranches) that are subject to default but *collectively* protected by exogenous collateral requirements. Within each family $k \in \mathcal{J}^t$ there is an priority as to the payment of the returns. We assume for the sake of simplicity that the collateral requirements for each $k \in \mathcal{J}^t$ consist solely of consumption goods (Physical Collateral) or of assets in $\mathcal{J}^b$.

We now explain the characteristics of and the rules for trading in each of the aforementioned asset types:

[1]The structure of real securities also allows the study of models with nominal securities, since Geanakoplos and Mas-Collel (1989) show that a model with securities that deliver nominal returns can be converted into a family of models with real securities.

2.4.1 The set $\mathcal{J}^b$: Assets with Bounded Short-Sales

Each asset $j \in \mathcal{J}^b$ is characterized by

1. A node $\xi_j \in \mathcal{D}$ where it is issued and a set of *terminal nodes*, $T_j \subset \mathcal{D}(\xi_j)$.

   We suppose that if $T_j \neq \phi$ then, given nodes μ_1, $\mu_2 \in T_j$, we have $\mathcal{D}(\mu_1) \cap \{\mu_2\} = \phi$.[2]

2. The sub-tree where the asset is negotiated

$$\mathcal{N}(j) = \{\xi \in \mathcal{D}(\xi_j) : \xi < \kappa \quad, \kappa \in T_j\},$$

3. The processes $A(\xi, j)$ of bundle promises, which are defined in the sub-tree of nodes where the asset gives returns, $\mathcal{R}(j) = \{\xi \in \mathcal{D}(\xi_j) : \xi_j < \xi \leq \kappa \quad, \kappa \in T_j\}$, and,

4. Real numbers $\left(b^j_\xi\right)_{\xi \in \mathcal{N}(j)} \in \mathbb{R}^{\mathcal{N}(j)}_{++}$ that bound the number of units of the asset j that each agent can sell at node ξ.

Thus, given an asset $j \in \mathcal{J}^b$ and a node $\xi \in \mathcal{N}(j)$, each agent $h \in \mathcal{H}(\xi)$ chooses a portfolio $Z^h(\xi, j) = \left(\theta^h(\xi, j) - \varphi^h(\xi, j)\right) \in \mathbb{R}$ in the asset, subject to the condition $\varphi^h(\xi, j) \leq b^j_\xi$; here we denote by $\theta^h(\xi, j) = \max\{Z^h(\xi, j), 0\}$ the number of units of the asset j in ξ bought, and by $\varphi^h(\xi, j) = -\min\{Z^h(\xi, j), 0\}$ the number of units sold.[3]

Such a portfolio costs

$$\text{PV}_{\xi,j}(\theta^h, \varphi^h) \equiv q_{\xi,j} \left(\theta^h(\xi, j) - \varphi^h(\xi, j)\right), \tag{1}$$

(where $q_{\xi,j}$ denotes the asset's unit price at node ξ), and delivers, at each node $\mu \in \xi^+$, returns equal to

$$\text{R}_{\mu,j}(\theta^h, \varphi^h) \equiv (p_\mu A(\mu, j) + q_{\mu,j}) \left(\theta^h(\xi, j) - \varphi^h(\xi, j)\right). \tag{2}$$

The next condition, which guarantees that the real returns of assets in $\mathcal{J}^b$ do not grow unboundedly along the tree, will be necessary for existence of equilibrium; this condition is commonplace in the literature of sequential equilibrium in infinite-horizon economies with multi-period asset

[2]This technical condition aims to avoid ambiguity when dealing with the elements of T_j.

[3]In the case of assets in $\mathcal{J}^b$, the agents have no interest in buying and selling units of the same type at the same time. On the other hand, in the case of assets that are subject to default, and are protected by collateral requirements, there will be distinct buy and sell prices as well as distinct return rates for borrowers and for lenders. Such spreads may motivate agents to buy and sell at the same time assets of the same type in order to obtain riskless future gains. We shall show below that in equilibrium this pathology does not occurr, and moreover that one may assume that, in each asset market, the agents behave either exclusively as borrowers or exclusively as lenders.

trading (see Magill and Quinzii (1996)):

ASSUMPTION A: For each asset $j \in \mathcal{J}^b$, the real promises $(A(\xi, j))_{\xi \in \mathcal{R}(j)}$ belong to $l^\infty(\mathcal{L} \times \mathcal{R}(j))$.

We denote by $\mathcal{J}^b(\xi)$ the (finite) set of assets $j \in \mathcal{J}^b$ which are traded at node ξ.

2.4.2 The set $\mathcal{J}^d$: Assets protected by Exogenous Collateral

Each asset $j \in \mathcal{J}^d$ is characterized by the node $\xi_j \in \mathcal{D}$ where it is issued; the set of *terminal nodes*, $T_j \subset \mathcal{D}(\xi_j)$; the sub-tree where the asset is *potentially* negotiated, [4]

$$\mathcal{N}(j) = \{\xi \in \mathcal{D}(\xi_j) : \xi < \kappa \quad, \kappa \in T_j\};$$

and the processes $A(\xi, j)$ of bundle promises, which are defined in the set of nodes where the asset delivers returns, $\mathcal{R}(j) = \{\xi \in \mathcal{D}(\xi_j) : \xi_j < \xi \leq \kappa \quad, \kappa \in T_j\}$.

If the asset j in $\mathcal{J}^d$ is negotiated at the node $\xi \in \mathcal{D}$ then we denote by $q_{\xi,j}$ the unit price of the asset at this node, associated with the process $[A(\mu, j)]_{\mu \geq \xi}$. Such prices may vary along the tree $\mathcal{N}(j)$ according to the rules for renegotiation that will be set out in the next section.

Moreover, there exist, at each node $\xi \in \mathcal{N}(j)$, collateral requirements $C_{\xi,j} = [C^1_{\xi,j}, C^2_{\xi,j}] \in \mathbb{R}^L_+ \times \mathbb{R}^{\mathcal{J}^b(\xi) \cup \mathcal{J}^d(\xi)}_+$, that depend on the price level and are given by the functions

$$C^1_{\xi,j} : \mathbb{R}^L_+ \times \mathbb{R}^{\mathcal{J}^b(\xi) \cup \mathcal{J}^d(\xi)}_+ \times [0,1]^{\mathcal{J}^d(\xi)}_+ \quad \rightarrow \quad \mathbb{R}^{\mathcal{L}}_+, \tag{3}$$

$$C^1_{\xi,j} : \mathbb{R}^L_+ \times \mathbb{R}^{\mathcal{J}^b(\xi) \cup \mathcal{J}^d(\xi)}_+ \times [0,1]^{\mathcal{J}^d(\xi)}_+ \quad \rightarrow \quad \mathbb{R}^{\mathcal{J}^b(\xi) \cup \mathcal{J}^d(\xi)}_+, \tag{4}$$

where $C^1_{\xi,j}(p_\xi, q_\xi; \beta_\xi)$ (resp. $C^2_{\xi,j}(p_\xi, q_\xi; \beta_\xi)$) represents the physical (resp. financial) collateral requirements for the asset $j \in \mathcal{J}^d$ at the node $\xi \in \mathcal{N}(j)$, and depends on the commodity price p_ξ at this node, and on the asset prices $q_{\xi,j'}$ and anonymous renegotiation rules $\beta_{\xi,j'}$ of the assets $j' \in \mathcal{J}^b(\xi) \cup \mathcal{J}^d(\xi)$. These rules, which are taken as given by the agents, are explained in the next section.

As usual, for each node ξ the set $\mathcal{J}^d(\xi)$ denotes the finite set of assets in $\mathcal{J}^d$ that are negotiated in the economy at node ξ.

For notational convenience, we refer to the collateral requirements at the node ξ for the asset $j \in \mathcal{J}^d$ as $C_{\xi,j} = [C^1_{\xi,j}, C^2_{\xi,j}]$.

[4] Since we allow the possibility of default along the event-tree, the assets might foreclose before the original terminal dates $\kappa \in T_j$.

As in Dubey, Geanakoplos, and Zame (1995) we allow the existence of penalties $\lambda^h_{\xi,j}$ for each node $\xi \in (\mathcal{D}^h - \{\xi^h\})$ and asset $j \in \mathcal{J}^d(\xi^-)$; such penalties correspond to the loss of utility for each unit of default incurred by agent h at the various nodes ξ and assets j.

It is worth noting that in this model the collateral requirements for the assets $j \in \mathcal{J}^d$ will consist of either commodities (*physical collateral*) or assets (*financial collateral*), or both. The physical collateral for the asset $j \in \mathcal{J}^d$ at the node $\xi \in \mathcal{N}(j)$ can be divided, as in Dubey, Geanakoplos, and Zame (1995), among $C^1_{\xi,j} = C^{1,W}_{\xi,j} + C^{1,B}_{\xi,j} + C^{1,L}_{\xi,j}$, where $C^{1,W}_{\xi,j}$, $C^{1,B}_{\xi,j}$, $C^{1,L}_{\xi,j}$ denote respectively the part that is stored, the part that is held by the borrower and the part that is held by the lender.

The physical collateral is subject to depreciation and we denote by $Y_\mu \diamond C^1_{\xi,j} = Y^c_\mu\big(C^{1,B}_{\xi,j} + C^{1,L}_{\xi,j}\big) + Y^w_\mu C^{1,W}_{\xi,j}$ the depreciated bundle of collateral $C^1_{\xi,j}$ at the node $\mu \in \xi^+$.

We suppose that the following assumption holds:

ASSUMPTION C: For each asset $j \in \mathcal{J}^d$, the real promises $(A(\xi,j))_{\xi\in\mathcal{R}(j)}$ belong to $l^\infty(\mathcal{L} \times \mathcal{R}(j))$, and the functions $C^1_{\xi,j}$, $C^2_{\xi,j}$ are continuous in $\big(p_\xi, (q_{\xi,j'})_{j'\in\mathcal{J}^b(\xi)\cup\mathcal{J}^d(\xi)\cup\{j\}}\big)$ and different from zero, for every node $\xi \in \mathcal{N}(j)$.

We suppose that the commodities and the assets in $\mathcal{J}^b \cup \mathcal{J}^d$ that can be used as collateral for the asset $j \in \mathcal{J}^d$ at node ξ, are the same along the sub-tree $\mathcal{N}(j)$ and are independent of the price level. That is, given a commodity $l \in \mathcal{L}$, if $\Big(C^1_{\xi,j}(p,q,\beta)\Big)_l \neq 0$ then $\big(C^1_{\mu,j}(p',q',\beta')\big)_l \neq 0$ for all node $\mu \in \mathcal{N}(j)$ and prices (p',q',β'). Analogously, given an asset $j' \neq j$ in $\mathcal{J}^b(\xi) \cup \mathcal{J}^d(\xi)$, if $\Big(C^2_{\xi,j}(p,q,\beta)\Big)_{j'} \neq 0$ then $\big(C^2_{\mu,j}(p',q',\beta')\big)_{j'} \neq 0$ for all price vectors (p',q',β') and nodes $\mu \in \mathcal{N}(j)$.

Moreover, we suppose that given prices for the commodities and for the assets, the collateral requirements $C_{\xi,j} = [C^1_{\xi,j}, C^2_{\xi,j}]$ are uniformly bounded from above by a positive vector $\overline{C}_j \in \mathbb{R}^{\mathcal{L}}_+ \times \mathbb{R}^{\mathcal{J}^b(\xi_j)\cup\mathcal{J}^d(\xi_j)}_+$ that is independent of the node.

Now, in order to prevent two assets j_1 and j_2 from mutually protecting each other via collateral requirements, we impose a *pyramiding* structure on the assets in $\mathcal{J}^d$.[5]

ASSUMPTION D: The set $\mathcal{J}^d$ is a disjoint union of sets $\big(\mathcal{A}_k; k \geq 0\big)$, which are defined by the recursive rule:

$$\mathcal{A}_0 = \{j \in \mathcal{J}^d : \big(C^2_{\xi,j}\big)_{j'} \neq 0 \Rightarrow j' \in \mathcal{J}^b\},$$

$$\mathcal{A}_k = \left\{ j \in \mathcal{J}^d : \big(C^2_{\xi,j}\big)_{j'} \neq 0 \Rightarrow j' \in \left(\bigcup_{r=0}^{k-1} \mathcal{A}_r\right) \cup \mathcal{J}^b \right\} - \bigcup_{r=0}^{k-1} \mathcal{A}_r.$$

[5]For more details on this concept see Dubey, Geanakoplos and Zame (1996).

Note that Assumption C guarantees that the sets $\mathcal{A}_k$ are independent of the price level and of the nodes.

2.4.3 Renegotiation Rules for the Assets in $\mathcal{J}^d$.

The agents in the economy take as given "*anonymous renegotiation rules*" $\beta_{\xi,j} \in [0,1]$, for every asset $j \in \mathcal{J}^d$ and node $\xi \in \mathcal{R}(j)$.

So, given an asset $j \in \mathcal{A}_0$ and a node $\xi \in \xi_j^+$,

- If $\beta_{\xi,j} = 1$ then the agents expect to receive, for each unit of asset j bought by them at the node ξ_j, the bundle $A(\xi,j)$ at this node, and to negotiate their long-position at prices $q_{\xi,j}$, with collateral requirements $C_{\xi,j}$.
- If $\beta_{\xi,j} = 0$, then the agents suppose that the asset $j \in \mathcal{A}_0$ goes to default at this node, delivering the depreciated value of the collateral

$$\text{DCV}_{\xi,j} \equiv p_\xi Y_\xi \diamond C^1_{\xi_j,j} + \sum_{j' \in \mathcal{J}^b(\xi_j)} \text{R}_{\xi,j'}\left(\left(C^2_{\xi_j,j}\right)_{j'}, 0\right), \tag{5}$$

  as well as the foreclosure.
- Finally, if $\beta_{\xi,j} \in (0,1)$, then the agents expect to receive, for each unit of asset j bought at ξ_j, a percentage of the original promises, $\beta_{\xi,j}A(\xi,j)$, and a collateral proportional to the non-delivered promises, that is, $(1-\beta_{\xi,j})\text{DCV}_{\xi,j}$. Moreover, the asset suffers an automatic renegotiation and is transformed into a new asset with the promises process $[\beta_{\xi,j}A(\mu,j)]_{\{\mu>\xi\ ,\ \mu\in\mathcal{R}(j)\}}$ and new collateral requirements $[\beta_{\xi,j}C^1_{\xi,j}, \beta_{\xi,j}C^2_{\xi,j}]$. The agents therefore expect to sell their long-positions at prices $\beta_{\xi,j}q_{\xi,j}$.

For the sake of simplicity, we will keep the notation j for such new assets in $\mathcal{A}_0$ and modify their collateral and price processes, in the sub-tree with root ξ, to $\left[\beta_{\xi,j}C_{\mu,j}; \beta_{\xi,j}q_{\mu,j}\right)_{\{\mu>\xi\ ,\ \mu\in\mathcal{N}(j)\}}$.

The interpretation of $\beta_{\xi,j}$ is similar at the nodes $\xi \in \mathcal{N}(j)$ with $\tilde{t}(\xi) > \tilde{t}(\xi_j)+1$. In fact,

- If $\beta_{\xi,j} = 1$ then the agents expect that the asset $j \in \mathcal{A}_0$, *that was negotiated at the node* $\xi-$, delivers the bundle associated to its promises. That is, the agents receive the commodities bundle $\left(\prod_{\xi_j<\nu<\xi}\beta_{\nu,j}\right)A(\xi,j)$. If the node ξ is not a terminal node of j, $\xi \notin T_j$, then the agents also expect to sell their long-positions at prices $\left(\prod_{\xi_j<\nu<\xi}\beta_{\nu,j}\right)q_{\xi,j}$.
- In the case where $\beta_{\xi,j} = 0$ all lenders expect to receive the depreciated value of the collateral

associated to the asset j,[6]

$$\text{DCV}_{\xi,j} \equiv \left(\prod_{\xi_j<\nu<\xi} \beta_{\nu,j}\right)\left[p_\xi Y_\xi \diamond C^1_{\xi^-,j} + \sum_{j'\in\mathcal{J}^b(\xi^-)} \text{R}_{\xi,j'}\left(\left(C^2_{\xi^-,j}\right)_{j'},0\right)\right], \tag{6}$$

as well as the asset's foreclosure in the market.

- If $\beta_{\xi,j}\in(0,1)$ then the asset j delivers $\beta_{\xi,j}$ percent of the promises made at the node ξ^-, $\beta_{\xi,j}\left[\left(\prod_{\xi_j<\nu<\xi}\beta_{\nu,j}\right)A(\xi,j)\right]$, and delivers a proportional part of the depreciated value of the collateral constituted at the node ξ^-, $(1-\beta_{\xi,j})\text{DCV}_{\xi,j}$.

  In the case where ξ is not a terminal node of j, then j is renegotiated with prices $\left(\prod_{\xi_j<\nu\le\xi}\beta_{\nu,j}\right)q_{\xi,j}$, new collateral requirements $\left(\prod_{\xi_j<\nu\le\xi}\beta_{\nu,j}\right)[C^1_{\xi,j},C^2_{\xi,j}]$, and a new promises process given by $\left[\left(\prod_{\xi_j<\nu\le\xi}\beta_{\nu,j}\right)A(\mu,j)\right]_{\{\mu>\xi\ ,\ \mu\in\mathcal{R}(j)\}}$.

In sum, each unit of the asset $j\in\mathcal{A}_0$ bought by an agent $h\in\tilde{\mathcal{H}}(\xi)$ has value $\left(\prod_{\xi_j<\nu\le\xi}\beta_{\nu,j}\right)q_{\xi,j}$, at the node $\xi\in\mathcal{N}(j)$, and delivers, at each node $\mu\in\xi^+$, an amount

$$L_{\mu,j} \equiv \left[\beta_{\mu,j}\left(p_\mu\tilde{\beta}_{\xi,j}A(\mu,j)\right) + (1-\beta_{\mu,j})\text{DCV}_{\mu,j}\right] \tag{7}$$

and is sold at prices $\tilde{\beta}_{\mu,j}q_{\mu,j}$, where $\tilde{\beta}_{\xi,j}$ is the renegotiation rule at the node ξ with respect to the node ξ_j

$$\tilde{\beta}_{\xi,j} = \prod_{\xi_j<\nu\le\xi}\beta_{\nu,j}. \tag{8}$$

On the other hand, at each node $\xi\in\mathcal{D}$, agents h in $\tilde{\mathcal{H}}(\xi)$ are able to determine *endogenous renegotiations rules* $\left[\alpha^h_{\xi,j}\right]_{j\in\mathcal{J}(\xi^-)}$.

That is, for each unit of the asset $j\in\mathcal{A}_0$ sold by an agent $h\in\tilde{\mathcal{H}}(\xi)$, he borrow the amount $\tilde{\beta}_{\xi,j}q_{\xi,j}$ for his short positions, and he must establish a collateral $\tilde{\beta}_{\xi,j}[C^1_{\xi,j},C^2_{\xi,j}]$, whose value is

$$\text{CV}_{\xi,j} \equiv \tilde{\beta}_{\xi,j}\left[p_\xi C^1_{\xi,j} + \sum_{j'\in\mathcal{J}^b} q_{\xi,j'}\left(C^2_{\xi,j}\right)_{j'}\right]. \tag{9}$$

Moreover, he delivers, at each node $\mu\in\xi^+$, the amount

$$B^h_{\mu,j} \equiv \left[\alpha^h_{\mu,j}\left(p_\mu\tilde{\beta}_{\xi,j}A(\mu,j)\right) + (1-\alpha^h_{\mu,j})\text{DCV}_{\mu,j}\right] \tag{10}$$

where $\alpha^h_{\mu,j}$ belongs to the interval [0,1] and represents the endogenous renegotiation rule of the agent h for the asset $j\in\mathcal{A}_0$, at the node μ. The interpretation of the different values of $\alpha^h_{\mu,j}$ is analogous to that made for the rules $\beta_{\mu,j}$, but now from the borrower's point of view:

[6]If the asset j suffers default before the node ξ then the collateral clearly is equal to zero.

- If $\alpha^h_{\mu,j} = 1$ then the agent h delivers, for each unit of asset $j \in \mathcal{J}^d$ sold by him at the node ξ (remember that $\mu \in \xi^+$), the bundle $\tilde{\beta}_{\xi,j} A(\mu, j)$ at this node, and renegotiates his short position at prices $\alpha^h_{\xi,j} \tilde{\beta}_{\xi,j} q_{\mu,j}$.

- If $\alpha^h_{\mu,j} = 0$ then the agent h goes to default at this node in the asset $j \in \mathcal{J}^d$: he delivers, for each unit of the asset sold at the node ξ, the depreciated value of the collateral $\mathrm{DCV}_{\mu,j}$, and forecloses his short position in asset j.

- If $\alpha^h_{\mu,j} \in (0,1)$ then the agent h delivers, for each unit of asset j sold at the node ξ, a percentage of the promises, $\alpha^h_{\mu,j} \tilde{\beta}_{\xi,j} A(\mu, j)$, he also expects that the lenders seize a proportional part of the collateral constituted at the node ξ,

$$(1 - \alpha^h_{\mu,j}) \tilde{\beta}_{\xi,j} [Y_\mu \diamond C^1_{\xi,j} + \sum_{j' \in \mathcal{J}^b} A(\mu, j') \left(C^2_{\xi,j} \right)_{j'}, C^2_{\xi,j}],$$

  and, in the case where μ is not a terminal node of the asset, he sells his short position with unit price $\alpha^h_{\mu,j} \tilde{\beta}_{\xi,j} q_{\mu,j}$.

Therefore, the portfolio $(\theta^h(\xi, j), \varphi^h(\xi, j))$ for the agent $h \in \tilde{\mathcal{H}}(\xi)$ in the asset $j \in \mathcal{A}_0$, at node ξ, is worth

$$\mathrm{PV}_{\xi,j}(\theta^h, \varphi^h) \equiv \mathrm{CV}_{\xi,j} \varphi^h(\xi, j) + \tilde{\beta}_{\xi,j} q_{\xi,j} \left(\theta^h(\xi, j) - \varphi^h(\xi, j) \right), \tag{11}$$

and delivers, at each node $\mu \in \xi^+$,

$$\mathrm{R}_{\mu,j}(\theta^h, \varphi^h) \equiv \left(L_{\mu,j} + \tilde{\beta}_{\mu,j} q_{\mu,j} \right) \theta^h(\xi, j) - \left(B^h_{\mu,j} + \alpha^h_{\mu,j} \tilde{\beta}_{\xi,j} q_{\mu,j} - \mathrm{DCV}_{\mu,j} \right) \varphi^h(\xi, j). \tag{12}$$

Now, for the assets in $j \in \bigcup_{k \neq 0} \mathcal{A}_k \subset \mathcal{J}^d$, the lenders also take as given anonymous renegotiation rules $\beta_{\xi,j}$, and the borrowers also choose endogenous renegotiation rules $\alpha^h_{\xi,j}$, at the nodes $\xi \in \mathcal{N}(j)$. For these assets, however, the depreciated value of collateral constituted at ξ^- depends on the returns of other assets in $\mathcal{J}^d$ as

$$\mathrm{DCV}_{\xi,j} \equiv \tilde{\beta}_{\xi^-,j} \left[p_\xi Y_\xi \diamond C^1_{\xi^-,j} + \sum_{j' \in \mathcal{J}^b(\xi^-) \cup \mathcal{J}^d(\xi^-)} \mathrm{R}_{\xi,j'} \left(\left(C^2_{\xi^-,j} \right)_{j'}, 0 \right) \right], \tag{13}$$

Note that Assumption D guarantees that equation (13) is well-defined. Therefore, analogously to the assets in $\mathcal{A}_0$, a portfolio $(\theta(\xi, j), \varphi(\xi, j))$ in the asset $j \in \mathcal{J}^d - \mathcal{A}_0$ is worth $\mathrm{PV}_{\xi,j}(\theta, \varphi)$, at the node ξ, and delivers, at each node μ, which is the immediate successor of the node ξ, a quantity $\mathrm{R}_{\mu,j}(\theta, \varphi)$, where the functions $\mathrm{PV}_{\xi,j}$ and $\mathrm{R}_{\mu,j}$ are defined by equations (11) and (12), using equation (13).

It is important to note that it is not necessary to identify secondary markets in the economy. That is, once every agent expects to receive returns as a function of the anonymous renegotiation rules $\beta_{\xi,j}$, which are given and are the same for all participants of the markets, then they are indifferent to the identity of the borrower.

Moreover, in some cases the endogenous renegotiation rules are the same for every agent. In fact, if we consider an asset whose returns[7] are less than its associated depreciated collateral at a node, then the optimal strategy for the agents able to negotiate this asset at this node is to take $\alpha^h_{\xi,j}$ equal to one, since other choices would diminish their wealth at this node.

Now, if there are no penalties for default, that is $\lambda^h \equiv 0$ for every agent h in $\mathcal{H}$, then the only enforcement in case of default is the collateral seizure by the lenders. So if the promises of the asset are greater than the associated collateral at a node ξ, then the optimal strategy is to take the value of $\alpha^h_{\xi,j}$ equal to zero, that is, going to default. Therefore, in the absence of these penalties, the only case in which the agents have different endogenous renegotiation rules is when the value of the collateral is the same as the face value.

We assume that in equilibrium there is a compatibility between the anonymous renegotiation rules $\beta_{\mu,j}$ and the endogenous rules $\alpha^h_{\mu,j}$, for each node $\mu \in \xi^+$ and asset j in $\mathcal{J}^d(\xi)$.

In fact, we require that the anonymous rule $\beta_{\mu,j}$ be a mean of the endogenous rules $\alpha^h_{\mu,j}$

$$\beta_{\mu,j} = \sum_{h \in \tilde{\mathcal{H}}(\xi)} \alpha^h_{\mu,j} p^h(\xi, j). \tag{14}$$

where $p^h(\xi, j)$ is the percentage of the total short-position on asset j owned at the node ξ by the agent h.

This implies that, in equilibrium, the total return delivered by an asset j will be equal to the total expected return, that is [8]

$$\sum_{h \in \tilde{\mathcal{H}}(\xi)} \left(L_{\mu,j} + \tilde{\beta}_{\mu,j} q_{\mu,j} \right) \overline{\theta}^h(\xi, j) = \sum_{h \in \tilde{\mathcal{H}}(\xi)} \left(B^h_{\mu,j} + \alpha^h_{\mu,j} \tilde{\beta}_{\xi,j} q_{\mu,j} \right) \varphi^h(\xi, j), \tag{15}$$

where $\overline{\theta}^h(\xi, j)$ denotes the total long position of the agent h, in the asset $j \in \mathcal{J}^d$, at the node ξ,

$$\overline{\theta}^h(\xi, j) \equiv \theta^h(\xi, j) + \sum_{j' \in \mathcal{J}^d(\xi)} \tilde{\beta}_{\xi,j'} \left(C^2_{\xi,j'} \right)_j \varphi^h(\xi, j'), \tag{16}$$

[7] At a node ξ the returns of the asset $j \in \mathcal{J}^d(\xi^-)$ are given by the value of real promises $\tilde{\beta}_{\xi^-,j} A(\xi, j)$ plus the spot price $\tilde{\beta}_{\xi^-,j} q_{\xi,j}$. If ξ is a terminal node for j, then the spot price is taken as zero.

[8] Note that equation (15) is a consequence of both the usual market clearing condition in equilibrium, $\sum_{h \in \tilde{\mathcal{H}}(\xi^-)} \overline{\theta}^h(\xi^-, j) = \sum_{h \in \tilde{\mathcal{H}}(\xi^-)} \varphi^h(\xi^-, j)$, and the fact that the rule $\beta_{\xi,j}$ is not a variable chosen by the agents.

and $\theta^h(\xi, j)$ (resp. $\varphi^h(\xi, j)$) is the *collateral-free* long-position (resp. short-position) in the asset j [9] of the agent h, at the node ξ.

2.4.4 The set $\mathcal{J}^t$: Short-Lived Tranches protected by Collateral

There are other financial structures in the markets which are protected by collateral requirements, such as the assets $j \in \mathcal{J}^d$, but which do not form "pyramids" with the latter. In fact, the assets $j \in \mathcal{J}^d$ may cause a chain of defaults if their promises are not honored: that is, default on an asset $j \in \mathcal{A}_k$ might cause defaults on all those assets in $\mathcal{A}_{k+1}$ that use it as collateral, and in their turn these might cause defaults on assets in $j \in \mathcal{A}_{k+2}$, and so forth.

One way of preventing this, while at the same time having a small amount of physical goods and assets protect (at least indirectly) all of the economy's assets which are subject to default, is to allow families of assets to be protected by collective collateral requirements. In this way the agents avoid having to constitute excessive collateral requirements, and they are also able to protect several promises with a single collateral requirement.

Consider for instance two assets j_1 and j_2 in $\mathcal{J}^d$, both of which are individually protected, at the node ξ, by the same collateral requirement $C_\xi \in \mathbb{R}^{\mathcal{L}}_+$, which is independent of the price level. Thus, if a given agent wishes to short sell a unit of j_1 and a unit of j_2, then he will have to twice constitute the bundle of goods C_ξ; this would entail very high costs. In this section we will study the scenario where the agent can protect both promises *jointly*, with the same collateral bundle C_ξ, but giving priority to the asset j_1 over the asset j_2 when it comes to paying the returns. In doing so, the agent may receive a smaller amount, since the buyers of asset j_2 will be more exposed to default. The agent, however, might improve her situation relative to the scenario where he sells either j_1 or j_2 but not both.

More formally, we suppose for simplicity that there exist only *short-lived tranches* along the event-tree $\mathcal{D}$. The set of tranches issued at the node ξ is given by a set $\mathcal{J}^t(\xi)$, where each $k \in \mathcal{J}^t(\xi)$ is characterized by a family of short-lived real assets $\{j_k^1, j_k^2, \ldots, j_k^{n_k}\}$ which make individual promises $A(\mu, j_k^m)_{m\in\{1,\ldots,n_k\}}$ at the immediate successor nodes $\mu \in \xi^+$, and are jointly protected by collateral requirements $C^k = [C_k^1, C_k^2]$. Such collateral requirements depend on the price level, and are given by the functions

$$C_k^1 : \mathbb{R}^{\mathcal{L}}_+ \times \mathbb{R}^{\mathcal{J}^b(\xi)}_+ \times \mathbb{R} \to \mathbb{R}^{\mathcal{L}}_+, \tag{17}$$

$$C_k^2 : \mathbb{R}^{\mathcal{L}}_+ \times \mathbb{R}^{\mathcal{J}^b(\xi)}_+ \times \mathbb{R} \to \mathbb{R}^{\mathcal{J}^b(\xi)}_+, \tag{18}$$

[9] In the sense that it does not involve assets backed as collateral.

where, analogously to the collateral requirements for assets in $\mathcal{J}^d$, $C_k^1(p_\xi, q_\xi, q_{\xi,k})$ (resp. $C_k^2(p_\xi, q_\xi, q_{\xi,k})$) denotes the amount of physical collateral (resp. financial collateral) that the borrowers of the tranche $k \in \mathcal{J}^t(\xi)$ must constitute for each unit that they sell. These amounts depend on the prices of the consumption goods at the node ξ, on the prices of the assets in $\mathcal{J}^b(\xi)$, and on the sale price of tranche k itself, $q_{\xi,k}$.

That is, even though the assets j_k^m that constitute the family $k \in \mathcal{J}^t(\xi)$ are short-lived, they may be protected both by physical goods and by multiperiod assets in $\mathcal{J}^b(\xi)$.

Again for the sake of simplicity we denote such requirements by $C^k = [C_k^1, C_k^2]$, recalling nevertheless that such baskets may vary with price changes.

We assume also that the physical collateral requirements can be broken down into $C_k^1 = C_k^{1,B} + C_k^{1,W}$, where $C_k^{1,B}$ denotes the portion of the collateral that is consumed by the borrower and $C_k^{1,W}$ denotes the portion that must be stored. We assume that no part of the collateral may be consumed by the lender; since each asset j_k^m may be separately negotiated by the lenders in the market, this assumption avoids defining a collateral assignment rule among the various buyers of the assets in k.

We impose the following conditions, which guarantee that the collateral requirements for each tranche vary continuously with the price levels:

ASSUMPTION E: For each tranche $j \in \mathcal{J}^t(\xi)$, the functions C_k^1, C_k^2 are continuous in $\left(p_\xi, (q_{\xi,a})_{a \in \mathcal{J}^b(\xi) \cup \{k\}}\right)$ and different from zero.

We denote by $Y_\mu \diamond C_k^1 \equiv Y_\mu^c C_k^{1,B} + Y_\mu^w C_k^{1,W}$ the depreciated value at the node $\mu \in \xi^+$ of the physical collateral constituted at ξ. As usual, we assume that the financial collateral C_2^k suffers no depreciation.

Now, at the immediate successor nodes $\mu \in \xi^+$ the asset j_k^m has priority over the assets $(j_k^r)_{r>m}$ as regards payments of the returns. *We suppose, for simplicity, that there do not exist utility penalties for agents who default.*

Therefore, each borrower of one unit of the tranche k delivers, at a node $\mu \in \xi^+$, the minimum between the depreciated value of the collateral

$$\mathrm{DCV}_{\mu,k} \equiv p_\mu Y_\mu \diamond C_k^1 + \sum_{j \in \mathcal{J}^b(\xi)} \mathrm{R}_{\mu,j}\left(\left(C_k^2\right)_j, 0\right) \tag{19}$$

and the total value of the promises made by the assets $\{j_k^1, j_k^2, \ldots, j_k^{n_k}\}$. That is,

$$B_{\mu,k} \equiv \min\left\{\mathrm{DCV}_{\mu,k}\,;\ \sum_{m=1}^{n_k} p_\mu A(\mu, j_k^m)\right\}. \tag{20}$$

On the other hand, the lenders are able to trade each asset $\{j_k^m\}_{\{k\in\mathcal{K},\ m\in\{1,\ldots,n_k\}\}}$. So for each unit of the asset j_k^m bought by a lender, he expects to receive, at each node $\mu \in \xi^+$, the amount

$$L_{\mu,k}^{j_k^1} \equiv \min\left\{p_\mu A(\mu, j_k^1)\,;\,\mathrm{DCV}_{\mu,k}\right\} \tag{21}$$

$$L_{\mu,k}^{j_k^m} \equiv \left[\min\left\{\sum_{i=1}^{m} p_\mu A(\mu, j_k^i)\,;\,\mathrm{DCV}_{\mu,k}\right\} - \sum_{i=1}^{m-1} p_\mu A(\mu, j_k^i)\right]^+, \quad m > 1. \tag{22}$$

We denote by q_{ξ,j_k^m} the price of asset $j_k^m \in k$ at the node ξ, and we assume that the sale price of a tranche k at node ξ is given by $q_{\xi,k} \equiv \sum_{m=1}^{n_k} q_{\xi,j_k^m}$.[10]

Hence, each agent $h \in \tilde{\mathcal{H}}(\xi)$ gets to choose a portfolio $\left[\left(\theta^h(\xi, j_k^m)\right)_{\{m=1,2,\ldots,n_k\}}\,;\varphi^h(\xi,k)\right]$ in tranche $k \in \mathcal{J}^t(\xi)$, paying an amount equal to

$$\mathrm{PV}_{\xi,k}(\theta^h, \varphi^h) \equiv \left[p_\xi C_k^1 + \sum_{j\in\mathcal{J}^b(\xi)} q_{\xi,j}\left(C_k^2\right)_j - q_{\xi,k}\right]\varphi^h(\xi,k) + \sum_{m=1}^{n_k} q_{\xi,j_k^m}\theta^h(\xi, j_k^m). \tag{23}$$

At each node $\mu \in \xi^+$ this portfolio yields returns equal to

$$\mathrm{R}_{\mu,k}(\theta^h, \varphi^h) \equiv \mathrm{DCV}_{\mu,k}\varphi^h(\xi,k) + \sum_{m=1}^{n_k} L_{\mu,k}^{j_k^m}\theta^h(\xi, j_k^m) - B_{k,\mu}\varphi^h(\xi,k). \tag{24}$$

We thus characterize the economy $\mathcal{E}\left(\mathcal{D},\mathcal{C},\mathcal{H},\mathcal{J}\right)$ by specifying the structure of uncertainty $\mathcal{D}$, the set of durable goods $\mathcal{C} = \left(\mathcal{L}, Y_\xi^c, Y_\xi^w\right)$, the characteristics of the agents $\mathcal{H} = \left(\mathcal{H}(\xi), \tilde{\mathcal{H}}(\xi), w^h, U^h\right)$, and the financial asset structure $\mathcal{J} = \left(\mathcal{J}^b, \mathcal{J}^d, \mathcal{J}^t\right)$.

2.5 Equilibrium

For notational convenience let us define the following sets:

- Assets that are bought at node $\xi \in \mathcal{D}$:

$$\mathcal{J}_B(\xi) = \mathcal{J}^b(\xi) \cup \mathcal{J}^d(\xi) \cup \left(\bigcup_{k\in\mathcal{J}^t(\xi)} \{j_m : j_m \in k\}\right),$$

- Assets and tranches that are sold at node $\xi \in \mathcal{D}$: $\mathcal{J}_S(\xi) \equiv \mathcal{J}^b(\xi) \cup \mathcal{J}^d(\xi) \cup \mathcal{J}^t(\xi)$,

- Assets (indexed by nodes) that can be bought by the agent $h \in \mathcal{H}$:

$$\mathcal{D}_B^h(\mathcal{J}) \equiv \left\{(\xi, a) : \xi \in \mathcal{D}^h\ , a \in \mathcal{J}_B(\xi)\right\},$$

[10] We are therefore interested in the existence of equilibrium in the case where the prices of the assets $j_m \in k$ are not perturbed relative to the tranche price $q_{\xi,k}$. This hypothesis is a good approximation of what occurrs in markets where there are no significant transaction costs for buying each asset j_k^m separately.

- Assets and tranches (indexed by nodes) that can be sold by the agent $h \in \mathcal{H}$:

$$\mathcal{D}^h_S(\mathcal{J}) \equiv \left\{(\xi, a) : \xi \in \mathcal{D}^h \ , a \in \mathcal{J}_S(\xi)\right\},$$

- Assets in $\mathcal{J}^d$ (indexed by nodes) that can be negotiated by the agent $h \in \mathcal{H}$:

$$\mathcal{D}^h(\mathcal{J}^d) \equiv \left\{(\xi, a) : \xi \in \mathcal{D}^h \ , a \in \mathcal{J}^d(\xi)\right\}.$$

Given a spot price process $(p_\xi)_{\{\xi \in \mathcal{D}\}}$, an asset price process $\left[(q_{\xi,j})_{\{\xi \in \mathcal{D}, j \in \mathcal{J}_B(\xi)\}}\right]$, and anonymous renegotiation rules $(\beta_{\xi,j})_{\{j \in \mathcal{J}^d, \xi \in \mathbb{R}(j)\}}$, each agent h in $\mathcal{H}$, able to trade in the sub-tree $\mathcal{D}^h$, can choose an allocation $(x, y, \theta, \varphi, \alpha)$ in the state-space
$\mathbb{E}^h = X^h \times Y^h \times \mathbb{R}_+^{\mathcal{D}^h_S(\mathcal{J})} \times \mathbb{R}_+^{\mathcal{D}^h_B(\mathcal{J})} \times [0,1]_+^{\mathcal{D}^h(\mathcal{J}^d)}$, subject to:

$$p_{\xi^h}\left[x(\xi^h) + y(\xi^h)\right] + \sum_{a \in \mathcal{J}_S(\xi^h)} \mathrm{PV}_{\xi^h,a}(\theta, \varphi) \leq p_{\xi^h} w^h(\xi^h). \tag{25}$$

For all $\xi > \xi^h \in \mathcal{D}^h - \delta\mathcal{D}^h$,

$$\begin{aligned} p_\xi\left[x(\xi) + y(\xi)\right] &+ \sum_{a \in \mathcal{J}_S(\xi)} \mathrm{PV}_{\xi,a}(\theta, \varphi) \\ &\leq p_\xi w^h(\xi) + p_\xi\left[Y^c_\xi x(\xi^-) + Y^w_\xi y(\xi^-)\right] + \sum_{a \in \mathcal{J}_S(\xi^-)} \mathrm{R}_{\xi,a}(\theta, \varphi). \end{aligned} \tag{26}$$

For every node $\xi \in \delta\mathcal{D}^h$,

$$p_\xi x(\xi) \leq p_\xi w^h(\xi) + p_\xi\left[Y^c_\xi x(\xi^-) + Y^w_\xi y(\xi^-)\right] + \sum_{a \in \mathcal{J}_S(\xi^-)} \mathrm{R}_{\xi,a}(\theta, \varphi), \tag{27}$$

and, for all $\xi \in \mathcal{D}^h - \delta\mathcal{D}^h$,

$$\varphi(\xi, j) \leq b^j_\xi, \quad \forall\, j \in \mathcal{J}^b(\xi). \tag{28}$$

Note that given an asset $j \in \mathcal{J}^b(\xi)$, an allocation $(x, y, \theta(\xi, j); \varphi(\xi, j))$ that satisfies the aforementioned restrictions is collateral-free, in the sense that such goods and assets are not used to protect other assets that are traded by the agent. We therefore proceed just as we did with assets $j \in \mathcal{J}^d$ (see equation (16)), and denote the total amount of asset j bought by agent h at node ξ by

$$\overline{\theta}(\xi, j) \equiv \theta(\xi, j) + + \sum_{j' \in \mathcal{J}^d(\xi)} \tilde{\beta}_{\xi,j'}\left(C^2_{\xi,j'}\right)_j \varphi(\xi, j') + \sum_{k \in \mathcal{J}^t(\xi)} \left(C^2_k\right)_j \varphi(\xi, k), \quad \forall j \in \mathcal{J}^b(\xi). \tag{29}$$

The total amount of goods demanded for consumption by agent $h \in \mathcal{H}$ at node ξ is given by

$$\overline{x}(\xi) \equiv x(\xi) + \sum_{j \in \mathcal{J}^d(\xi)} \tilde{\beta}_{\xi,j} \left(C^{1,B}_{\xi,j} + C^{1,L}_{\xi,j} \right) \varphi(\xi, j) + \sum_{k \in \mathcal{J}^t(\xi)} C^{1,B}_k \varphi(\xi, k). \tag{30}$$

And the total amount of goods demanded for storage is given by

$$\overline{y}(\xi) \equiv y(\xi) + \sum_{j \in \mathcal{J}^d(\xi)} \tilde{\beta}_{\xi,j} C^{1,W}_{\xi,j} \varphi(\xi, j) + \sum_{k \in \mathcal{J}^t(\xi)} C^{1,W}_k \varphi(\xi, k). \tag{31}$$

Now, given prices and renegotiation rules (p, q, β), we define the budget set of agent h as

$$\mathcal{B}^h(p, q, \beta) = \left\{ (x, y, \theta, \varphi, \alpha) \in I\!\!E^h : \text{equations (25), (26) and (27) hold} \right\}. \tag{32}$$

Because our objective is to show the existence of equilibrium, we restrict the space of prices to $\mathcal{P} \equiv \{(p_\xi, q_\xi)_{\xi \in \mathcal{D}} : (p_\xi, q_\xi) \in \Delta_+^{\mathcal{L} + \mathcal{J}_B(\xi) - 1}\}$, where Δ_+^{n-1} denotes the n-dimensional simplex.

Note that given an allocation $(x^h, y^h, \theta^h, \varphi^h, \alpha^h) \in \mathcal{B}^h(p, q, \beta)$, the associated consumption at a node $\xi \in \mathcal{D}^h$ is [11]

$$c^h_\xi(x^h, y^h, \theta^h, \varphi^h) \equiv \overline{x}^h(\xi) - \sum_{j \in \mathcal{J}^d(\xi)} \tilde{\beta}_{\xi,j} \left[C^{1,L}_{\xi,j} \varphi^h(\xi, j) - C^{1,L}_{\xi,j} \overline{\theta}^h(\xi, j) \right], \tag{33}$$

and the *penalty for default* at ξ is

$$\begin{aligned} P^h_{(\xi, \varphi^h, \alpha^h)} &\equiv \sum_{j \in \mathcal{J}^d(\xi^-)} \lambda^h_{\xi,j} \left[p_\xi \tilde{\beta}_{\xi^-,j} A(\xi, j) + \tilde{\beta}_{\xi^-,j} q_{\xi,j} - \left(\alpha^h_{\xi,j} \tilde{\beta}_{\xi^-,j} q_{\xi,j} + B^h_{\xi,j} \right) \right]^+ \varphi^h(\xi^-, j) \\ &\equiv \sum_{j \in \mathcal{J}^d(\xi^-)} \lambda^h_{\xi,j} \left(1 - \alpha^h_{\xi,j} \right) \left[p_\xi \tilde{\beta}_{\xi^-,j} A(\xi, j) + \tilde{\beta}_{\xi^-,j} q_{\xi,j} - \mathrm{DCV}_{\xi,j} \right]^+ \varphi^h(\xi^-, j) \end{aligned}$$

It is useful, to shorten the notations, define the objective function of the agent h, evaluated in an allocation $(x, y, \theta, \varphi, \alpha)$ in the budget set $\mathcal{B}^h(p, q, \beta)$, by

$$V^h(x, y, \theta, \varphi, \alpha) \equiv U^h \left[\left(c^h_\xi(x, y, \theta, \varphi) \right)_{\xi \in \mathcal{D}^h} \right] - \sum_{\xi \in \mathcal{D}^h} P^h_{(\xi, \varphi, \alpha)}. \tag{34}$$

It follows that given prices and anonymous renegotiation rules $(p, q, \beta) \in \mathcal{P} \times [0,1]^{\mathcal{D}^h(\mathcal{J}^d)}$, then consumer h's problem in the economy $\mathcal{E}$ is

$$\begin{aligned} &\max && V^h(x, y, \theta, \varphi, \alpha), \\ &\text{subject to} && (x, y, \theta, \varphi, \alpha) \in \mathcal{B}^h(p, q, \beta). \end{aligned} \tag{35}$$

We now define the three possible notions of equilibrium that arise from the alternative definitions for budget-set:

[11] Throughout the paper we use the convention that given a node ξ, then $\varphi^h(\xi) = \overline{\theta}^h(\xi) = 0$ for the agents $h \notin \tilde{\mathcal{H}}(\xi)$.

Definition 1 *An Equilibrium for the Infinite Horizon Exogenous Collateral Economy* $\mathcal{E}\left(\mathcal{D},\mathcal{C},\mathcal{H},\mathcal{J}\right)$ *is a vector of prices and anonymous renegotiation rules* $(p,q,\beta)\in\mathcal{P}\times[0,1]^{\mathcal{D}(\mathcal{J})}$ *such that there exist allocations* $\left[x^h,y^h,\theta^h,\varphi^h,\alpha^h\right]_{h\in\mathcal{H}}\in\Pi_{h\in\mathcal{H}}\mathbb{E}^h$ *that satisfy*

I. For each agent $h\in\mathcal{H}$*, the allocation* $(x^h,y^h,\theta^h,\varphi^h,\alpha^h)$ *solves the problem (35) in* $\mathcal{B}^h(p,q,\beta)$ *.*

II. Asset markets are cleared, that is, for each node ξ *and asset* $j\in\mathcal{J}^b(\xi)\cup\mathcal{J}^d(\xi)$*. That is, if* $j\in\mathcal{J}^b(\xi)$ *or* $\tilde{\beta}_{\xi,j}\neq 0$ *we have*

$$\sum_{h\in\tilde{\mathcal{H}}(\xi)}\overline{\theta}^h(\xi,j)=\sum_{h\in\tilde{\mathcal{H}}(\xi)}\varphi^h(\xi,j). \tag{36}$$

III. Tranche markets are cleared, that is, for each node ξ *and tranche* $k\in\mathcal{J}^t(\xi)$ *we have*

$$\sum_{h\in\tilde{\mathcal{H}}(\xi)}\theta^h(\xi,j_m)=\sum_{h\in\tilde{\mathcal{H}}(\xi)}\varphi^h(\xi,k),\quad\forall j_k^m\in k. \tag{37}$$

IV. The total demand for commodities is equal to the total wealth at each node of the economy

$$\sum_{h\in\mathcal{H}(\xi_0)}\overline{x}^h(\xi_0)+\sum_{h\in\mathcal{H}(\xi_0)}\overline{y}^h(\xi_0)=\sum_{h\in\mathcal{H}(\xi)}w^h(\xi_0), \tag{38}$$

$$\sum_{h\in\mathcal{H}(\xi)}\overline{x}^h(\xi)+\sum_{h\in\tilde{\mathcal{H}}(\xi)}\overline{y}^h(\xi)=\sum_{h\in\mathcal{H}(\xi)}w^h(\xi)+\sum_{h\in\tilde{\mathcal{H}}(\xi^-)}\left[Y_\xi^c\overline{x}^h(\xi^-)+Y_\xi^s\overline{y}^h(\xi^-)\right]. \tag{39}$$

V. The endogenous renegotiation rules perfect-foresight the anonymous rules. In other words, for each node ξ *and asset* $j\in\mathcal{J}^d(\xi^-)$ *such that* $\tilde{\beta}_{\xi^-,j}\neq 0$*, we have*

$$\sum_{h\in\tilde{\mathcal{H}}(\xi^-)}\beta_{\xi,j}\overline{\theta}^h(\xi^-,j)=\sum_{h\in\tilde{\mathcal{H}}(\xi^-)}\alpha_{\xi,j}^h\varphi^h(\xi^-,j). \tag{40}$$

3 Existence of Equilibrium

In this section we state the main results of the paper, that is, the theorems that guarantees the existence of equilibrium in the Exogenous Collateral Economy.[12]

[12]Nesta versão preliminar do artigo so apresentaremos os resultados de existencia de equilibrio no caso em que o unico enforcement em caso de default é dado pelo dereito dos credores de receber o valor depreciado dos requerimentos de colateral.

Theorem 1: *Suppose that Assumptions A to E are satisfied, and*

a. *There exists an scalar* $\underline{w}$ *such that, for each agent* h *in* $\mathcal{H}$, $||w^h(\xi)||_\Sigma \geq \underline{w}$ *for all node* ξ *in* $\mathcal{D}^h$. *Moreover, there exists an upper bound in the agregated initial endowments at each node of the economy, that is,* $\sum_{h\in\mathcal{H}(\xi)} w^h(\xi,l) \leq \overline{W}$ *for all* (ξ,l) *in* $\mathcal{D}\times\mathcal{L}$;

b. *The utility function for the agent* h, U^h, *is separable in the time and in the states of nature, in the sense that*

$$U^h(c) = \sum_{\xi\in\mathcal{D}^h} u^h(\xi, c(\xi)),$$

where the functions $u^h(\xi,\ \ )$ *are concave, continuous and increasing, with* $u^h(\xi,0)=0$;

c. *There not exists penalties for default* $\lambda^h_{\xi,j}$, *in the utility functions of the agents* $h\in\tilde{\mathcal{H}}(\xi)$, *for the assets* $j\in\mathcal{J}^d(\xi)$;

d. *The structure of depreciation* $[Y^c_\xi, Y^s_\xi]$ *is given by*

$$Y^c_\xi \equiv diag\left[(a_l(\xi))_{l\in\mathcal{L}}\right],\quad Y^w_\xi \equiv diag\left[(b_l(\xi))_{l\in\mathcal{L}}\right],$$

and there exists $\kappa\in(0,1)$ *such that the scalars* $a_l(\xi)$ *and* $b_l(\xi)$ *satisfies,*

$$\max\{a_l(\xi), b_l(\xi)\} \leq \kappa \quad, \forall\xi\in\mathcal{D};$$

e. *The collateral requirements for the assets in* $\mathcal{J}^d\cup\mathcal{J}^t$ *are bounded for below, node by node, independently of the price level. That is,* $||C_{\xi,j}(p,q,\beta)||_\Sigma \geq \underline{C}_{\xi,j}$;

f. *The real promisses of the assets in* $\mathcal{J}^b$ *are uniformly bounded for above along the event-tree:*

$$||A(\xi,j)||_\infty < \overline{A},\quad \forall j\in\mathcal{J}^b(\xi),\ \forall\xi\in\mathcal{D}.$$

Moreover, there exists a scalar $\overline{K}$ *such that the number of assets in* $\mathcal{J}^b$, *that are negotiated at the node* ξ, *satisfies:* $\#\mathcal{J}^b(\xi)\leq\overline{K}$, *for all* $\xi\in\mathcal{D}$. *The upper bounds for the short sales of the assets in* $\mathcal{J}^b$ *are uniformly bounded for above along the event-tree,* $b^j_\xi \leq \bar{b}$, $\forall j\in\mathcal{J}^b(\xi)$, $\forall\xi\in\mathcal{D}$;

then there exists an equilibrium in the economy $\mathcal{E}(\mathcal{D},\mathcal{C},\mathcal{H},\mathcal{J})$.

Proof: See Appendix A.

Some comments about hyphoteses:

Conditions [a.] and [b.] are traditional in the literature. Condition [d.] guarantees that the total agregated endowment at each node of the economy is uniformly bounded from above along the event-tree. This condition, that appear in Araujo, Páscoa and Torres-Martínez (2000), impose that the depreciation structure will be strong enough along the time.

Condition [e.] guarantees that, independently of the price level, the collateral requirements are bounded from below, at each node of the event-tree. Note that this is always satisfied in the case that collateral requirements does not depend on the prices, as in Dubey, Geanakoplos, and Zame (1995) model.

Condition [f.] imposes the existence of a finite number of default-free assets negotiated at each node of the economy. This assumption is very strong, since it implies the existence of a finite number of infinite-lived assets that are protected by financial collateral. The assumption is, nevertheless, required to guarantee that the agents cannot make Ponzi schemes through the negotiation of the assets in $\mathcal{J}^b$.

It would therefore be interesting to obtain equilibrium results which do not rely on a priori bound on the number of default-free asset that are negotiated in the economy. In this direction, if we still assume that the only enforcement in case of default is the seizure of the collateral by the lenders, then all of the agents who sell assets in $\mathcal{J}^d$ will deliver, as returns, the minimum between the value of the real promises and the depreciated value of the collateral constituted by them.

Hence a short position in such assets delivers non-negative returns in the following period (since the agents will have the depreciated value of the collateral and have to deliver the minimum between that amount and the value of the real promises).

Thus, if there are no assets in the economy other than those which are either subject to default or constitute collateral tranches, every borrower acts as a lender of assets which only they can buy; they therefore have no interest in becoming borrowers "at infinity".

Corollary 1: *Under the hypotheses [a.]-[e.] of Theorem 1, there is always an equilibrium if any of the following conditions is met:*

a. *There are no infinite-lived agents trading in the economy, or*

b. *The only assets traded in the economy are those belonging to* $\mathcal{J}^d \cup \mathcal{J}^t$.

Proof: See Appendix A.

It should be noted that the transversality conditions and the debt constraints guarantee the *generic* existence of equilibrium in default-free economies whose assets may deliver dividends along several periods, as in Magill and Quinzii (1996) or Levine and Zame (1996). Therefore, the imposition of collateral requirements in an economy with default is not only less ad-hoc than the imposition of debt restrictions, but also guarantees that there are prices which centralize the agents' decisions.

4 Asset Pricing and Speculative Bubbles

4.1 Admissible Prices

In this section, we will characterize the *admissible prices*, that is, the prices which give a finite optimum to the agents' problems. We suppose that Assumptions A to E are satisfied and that the hypothesis of Theorem 1 holds.

It is important to note that the characterization of non-arbitrage prices does not make sense in the context of exogenous collateral models, because the feasibility allocations are bounded, node by node, and this prevents the occurrence of any arbitrage opportunities through allocations that satisfy the feasibility conditions.

Notice that, in equilibrium, the prices $(\overline{p}, \overline{q})$ provide finite solutions for the agents' problems: this follows from both the uniform bound along the event-tree in the consumption allocations, and from the finiteness of the utilities in consumption processes that are bounded from above along the event-tree.

Therefore, given an agent h in $\mathcal{H}$, we are interested in the prices $(p, q) \in \mathcal{P}$ for which there exist anonymous renegotiation rules β that provide problem (2.41) with a finite solution. A necessary condition for this is that, at each node ξ in the event-tree, there do not exist allocations that give with certainty either real positive returns or an improvement of the utility, without any cost.

In other words, one necessary condition for the problem of consumer $h \in \mathcal{H}$ to have a finite solution at prices (p, q, β) is that no allocation $(x, y, \theta, \varphi, \alpha)$ satisfy the following conditions at node $\xi \in \mathcal{D}^h - \delta\mathcal{D}^h$:

$$p_\xi\big[x(\xi)+y(\xi)\big] + \sum_{a\in\mathcal{J}_S(\xi)} \mathrm{PV}_{\xi,a}(\theta,\varphi) \quad \leq \quad 0; \tag{41}$$

$$\big[c^h_\xi(x,y,\theta,\varphi)\big]_l \quad \geq \quad 0 \qquad ,\forall l \in \mathcal{L}; \tag{42}$$

$$W(\mu; x, y, \theta, \varphi, \alpha) \equiv p_\mu\left[Y^c_\mu x(\xi) + Y^w_\mu y(\xi)\right] + \sum_{a\in\mathcal{J}_S(\xi)} \mathrm{R}_{\mu,a}(\theta,\varphi) \quad \geq \quad 0 \qquad ,\forall \mu \in \xi^+; \tag{43}$$

$$\max_{\{x: p_\mu x \leq W(\mu; x,y,\theta,\varphi,\alpha)\}} \left[u^h(\mu, x) - P^h_{(\mu,\varphi,\alpha)}\right] \quad \geq \quad 0 \qquad ,\forall \mu \in \xi^+; \tag{44}$$

where at least one of the inequalities is strict.

In fact, if an allocation satisfies the equations above, with at least one strict inequality, then either the associated consumption bundle is non-trivial (equation (42)), or there is a node μ, which is an immediate successor of node ξ, at which the allocation delivers positive returns (equation (44)), without entailing costs at node ξ (equation (41)).

4.2 Asset Pricing in Economies without Utility Loss for Default

For the sake of simplicity we will analyze admissible prices (p, q) in the case where the agents' objective functions *include no penalties for default.*

In this particular case, the existence of an allocation $(x, y, \theta, \varphi, \alpha)$ that satisfies the above equations with at least one strict inequality, at the prices (p, q, β), is equivalent to the existence of an allocation (x, y, θ, φ) that satisfies, with at least one strict inequality, the following conditions:

$$p_\xi\big[x(\xi)+y(\xi)\big]+\sum_{a\in\mathcal{J}_S(\xi)}\mathrm{PV}_{\xi,a}(\theta,\varphi) \leq 0; \tag{45}$$

$$\big[c^h_\xi(x,y,\theta,\varphi)\big]_l \geq 0 \quad ,\forall l\in\mathcal{L}; \tag{46}$$

$$W(\mu;x,y,\theta,\varphi,\beta) \geq 0 \quad ,\forall \mu\in\xi^+; \tag{47}$$

In fact, if (x, y, θ, φ) satisfies the conditions (45)-(47) then $(x, y, \theta, \varphi, \beta)$ satisfies equations (41)-(44) (because we are assuming that there are no utility penalties for agents who default).

Conversely, if $(x, y, \theta, \varphi, \alpha)$ satisfies equations (41)-(44) then, since there are no utility losses for agent $h \in \mathcal{H}$ in case of default, we can assume without loss of generality that $\beta_{\mu,j} = 1$ if the value of the promises associated with asset j, at node μ, is *strictly less* than the depreciated value of the collateral constituted at ξ. If the depreciated value of the collateral is *strictly greater* than the value of the promises at μ then we can assume that $\beta_{\mu,j}$ equals zero, and if the promises and the depreciated collateral are worth the same then the value of $\beta_{\mu,j}$ may take any value in the interval $[0, 1]$. Hence allocation $(x, y, \theta, \varphi, \beta)$ also satisfies equations (41)-(44) and therefore equations (45)-(47).

This last remark allows us to study the functional form of the admissible prices at a node $\xi \in \mathcal{D}$, independent of the agents who trade at that node.

Note now that, given the prices of the commodities and of the assets in the economy, then for each node ξ the functions $\mathrm{PV}_{\xi,a}$ and $[\mathrm{R}_{\mu,a}]_{\mu\in\xi^+}$ are linear in (θ, φ) for every asset $a \in \mathcal{J}_S(\xi) \cup \mathcal{J}_B(\xi)$. In fact, for each $a \in \mathcal{J}^b(\xi) \cup \mathcal{J}^d(\xi)$ we have

$$\mathrm{PV}_{\xi,a}(\theta,\varphi) = \mathrm{PV}_{\xi,a}(1,0)\,\theta(\xi,a)+\mathrm{PV}_{\xi,a}(0,1)\,\varphi(\xi,a), \tag{48}$$

$$\mathrm{R}_{\mu,a}(\theta,\varphi) = \mathrm{R}_{\mu,a}(1,0)\,\theta(\xi,a)+\mathrm{R}_{\mu,a}(0,1)\,\varphi(\xi,a),\quad \forall\mu\in\xi^+. \tag{49}$$

And for each tranche $k \in \mathcal{J}^t(\xi)$ we have

$$
\begin{aligned}
(50) \qquad \mathrm{PV}_{\xi,k}(\theta,\varphi) &= \sum_{m=1}^{n_k} q_{\xi,j_k^m}\, \theta(\xi, j_k^m) + \mathrm{PV}_{\xi,a}\,(0;1)\ \varphi(\xi,k),\\
(51) \qquad \mathrm{R}_{\mu,k}(\theta,\varphi) &= \sum_{m=1}^{n_k} L_{\mu,k}^{j_k^m}\, \theta(\xi, j_k^m) + \mathrm{R}_{\mu,a}\,(0;1)\ \varphi(\xi,k), \quad \forall \mu \in \xi^+.
\end{aligned}
$$

Thus, the conditions given by equations (45)-(47) may be written in matrix form. That is, for every node ξ there is a matrix $\mathbb{A}_\xi$ such that, if the prices (p,q) are admissible then there is no allocation $(x,y,\theta,\varphi) \in \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{L}} \times \mathbb{R}_+^{\mathcal{J}_B(\xi)\times\mathcal{J}_S(\xi)}$ that satisfies[13]

$$
(52) \qquad \mathbb{A}_\xi \begin{pmatrix} x \\ y \\ \theta \\ \varphi \end{pmatrix} \geq 0.
$$

It follows from this, by applying Stiemke's lemma (see Hildenbrand (1974)), that if the optimal utility level of the consumers $h \in \mathcal{H}$ is *finite*, at the price level (p,q,β), then there are *strictly positive* state prices $\left[\gamma_\xi, (\tilde{\gamma}_{\xi,l})_{l\in\mathcal{L}}, (\gamma_{\xi,l})_{l\in\mathcal{L}}, (\gamma_{\xi,j}^L)_{j\in\mathcal{J}_B(\xi)-\mathcal{J}^b(\xi)}, (\gamma_{\xi,j}^B)_{j\in\mathcal{J}_S(\xi)-\mathcal{J}^b(\xi)}\right]_{\xi\in\mathcal{D}}$ such that, given a node $\xi \in \mathcal{D}$, we have

- For each commodity $l \in \mathcal{L}$,

$$
(53) \qquad p_{\xi,l} = \frac{1}{\gamma_\xi}\left(\sum_{\mu\in\xi^+} \gamma_\mu p_\mu (Y_\mu^c)_l + \tilde{\gamma}_{\xi,l}\right),
$$

- For each asset $j \in \mathcal{J}^b(\xi)$,

$$
(54) \qquad q_{\xi,j} = \frac{1}{\gamma_\xi}\left(\sum_{\mu\in\xi^+} \gamma_\mu \left(p_\mu A(\mu,j) + q_{\mu,j}\right)\right),
$$

- For each asset $j \in \mathcal{J}^d(\xi)$,

$$
\begin{aligned}
(55) \qquad \tilde{\beta}_{\xi,j} q_{\xi,j} &= \frac{1}{\gamma_\xi}\left(\sum_{\mu\in\xi^+} \gamma_\mu \left(L_{\mu,j} + \tilde{\beta}_{\mu,j} q_{\mu,j}\right) + \sum_{l\in\mathcal{L}} \gamma_{\xi,l} \tilde{\beta}_{\xi,j} (C_{\xi,j}^{1,L})_l + \gamma_{\xi,j}^L\right),\\
(56) \qquad \mathrm{CV}_{\xi,j} - \tilde{\beta}_{\xi,j} q_{\xi,j} &= \frac{1}{\gamma_\xi}\left(\sum_{\mu\in\xi^+} \gamma_\mu \left(\mathrm{DVC}_{\mu,j} - B_{\mu,j}^h - \tilde{\beta}_{\mu,j} q_{\mu,j}\right)\right)\\
&\quad + \frac{1}{\gamma_\xi}\left(\sum_{l\in\mathcal{L}} \gamma_{\xi,l} \tilde{\beta}_{\xi,j}\left(C_{\xi,j}^{1,B} + \sum_{j'\in\mathcal{J}^d(\xi)} C_{\xi,j'}^{1,L} \tilde{\beta}_{\xi,j'} (C_{\xi,j}^2)_{j'}\right)_l + \gamma_{\xi,j}^B\right),
\end{aligned}
$$

[13]The matrix $\mathbb{A}_\xi$ depends on the price level (p,q) and on the anonymous renegotiation rules.

- For each tranche $k \in \mathcal{J}^t(\xi)$,

$$q_{\xi,j_k^m} = \frac{1}{\gamma_\xi}\left(\sum_{\mu\in\xi^+}\gamma_\mu L_k^{j_k^m} + \gamma^L_{\xi,j_k^m}\right), \quad \forall m \in \{1,\ldots,n_k\}, \tag{57}$$

$$\mathrm{CV}_{\xi,k} - q_{\xi,k} = \frac{1}{\gamma_\xi}\left(\sum_{\mu\in\xi^+}\gamma_\mu\left(\mathrm{DCV}_{\mu,k} - B_{\mu,k}\right) + \gamma^B_{\xi,k}\right). \tag{58}$$

Hence, the admissibility of the prices (p, q) implies that there are state prices which allow us to express a given asset's sale and purchase prices, at a node ξ, as the discounted value of the future dividents generated by the asset.[14] Note that the functional form of the prices of assets in $\mathcal{J}^d \cup \mathcal{J}^t$ includes non-linearities, which follow from the positiveness conditions imposed on the purchase and sale portfolios.

Due to such non-linearities, which are given by strictly positive deflators, the purchase prices of all assets negotiated at a node ξ will be strictly positive. Furthermore, since in our context there are no utility penalties for agents who default on an asset $j \in \mathcal{J}^d$, the sale prices of assets in $\mathcal{J}^d$ will be non-negative, that is

$$\mathrm{CV}_{\xi,j} - \tilde{\beta}_{\xi,j} q_{\xi,j} \geq 0, \quad \forall \xi \in \mathcal{D},\ \forall j \in \mathcal{J}^d(\xi). \tag{59}$$

This last inequality is fundamental for obtaining conditions which characterize the existence of speculative bubbles in the prices of assets subject to default.

4.2.1 Speculative Bubbles in Admissible Prices

In the literature on Sequential Equilibrium in Infinite-Horizon Economies, the existence of speculative bubbles in non-arbitrage prices has been studied, among others, by Santos and Woodford (1997) and Magill and Quinzii (1996), in the case where the agents cannot default on their promises and where short sales are bounded by debt restrictions or by exogenous transversality conditions.

In markets where the agents are required to constitute margin requirements when short selling, whether the aim is to protect buyers in case of future default, it is important to study speculation in admissible prices, given the possible applications to the macroeconomic literature on financial crises.

[14] For the sake of simplicity we will in this section treat commodities as assets, because their durability allows us to regard them as Lucas' trees; that is, as rights to benefit from part of a firm's output through the current consumption of certain bundles, as well as through the right to sell the depreciated values of such amounts in future states of nature. The non-linearity of commodity prices follows from the fact that agents cannot buy such "rights".

It is natural to expect that the admissible price of an asset backed by other assets or goods exhibits bubbles only if the admissible price of one of the corresponding collateral assets or goods differs from its fundamental value. In fact, one might think of an asset that is backed by a (financial or physical) collateral bundle as a derivative, whose underlying assets are those securities or goods that back the short sales. In this sense, one might intuitively think of distortions in a derivative's price relative to its fundamental value as always being caused by bubbles in the prices of one of the underlying assets.

In this section we will discuss such issues and show that the admissible price of an asset protected by collateral, ($j \in \mathcal{J}^d$), exhibits bubbles if and only if the following two conditions hold: the asset is negotiated, at these prices, for an infinite number of periods, and the collateral's depreciated value no tends to zero at infinity (see Lemma 3 below).

Using this result and the pyramiding structure given by Assumption D, we show that if an asset $j \in \mathcal{J}^d$ exhibits bubbles, then there is price speculation for at least one of the *default-free assets* in the economy, that is, those assets which belong to $\mathcal{L} \cup \mathcal{J}^b$. Thus, in order to prevent price distortions relative to the fundamental value of a given asset, it is enough to guarantee that there are no bubbles in the assets that are free of default. Conceivably, one way of doing so could be to adapt the results by Santos and Woodford (1996) and Magill and Quinzii (1996), since in an infinite-horizon economy without default every $j \in \mathcal{L} \cup \mathcal{J}^b$ can be regarded as an asset whose short sales are exogenously restricted (see Remark 2.2 below).

Since we are dealing with short-lived (i.e., single-period) tranches, the equations (57) - (58) guarantee that the sale price of tranche $k \in \mathcal{J}^t(\xi)$ as well as the purchase prices of the assets in family k do not exhibit speculation, because they are equal to the discounted value (using the γ deflators) of the future dividends.

We therefore restrict our attention to durable consumption goods and to assets belonging to $\mathcal{J}^b \cup \mathcal{J}^d$.

Given admissible prices (p, q) and renegotiation rules β, consider a state prices vector $\Gamma_\xi = \big[\gamma_\mu, (\tilde{\gamma}_{\mu,l})_{l\in\mathcal{L}}, (\gamma_{\mu,l})_{l\in\mathcal{L}}, (\gamma^L_{\mu,j})_{j\in\mathcal{J}_B(\mu)-\mathcal{J}^b(\mu)}, (\gamma^B_{\mu,j})_{j\in\mathcal{J}_S(\mu)-\mathcal{J}^b(\mu)}\big]_{\mu\in\mathcal{D}(\xi)}$ that satisfies equations (53)-(56) at each node $\mu \in \mathcal{D}(\xi)$.

It is important to note that since markets are incomplete, there might not be a unique state prices vector Γ_ξ at ξ. We therefore define the fundamental value of the various assets negotiated at node ξ as a function of the chosen state prices vector Γ_ξ.

• **Commodities.**

The *fundamental value* at node ξ of a durable good $l \in \mathcal{L}$ is given by

$$\text{(60)} \qquad \mathrm{F}(\xi, \Gamma_\xi, l) \equiv \frac{1}{\gamma_\xi} \left(\sum_{\mu \geq \xi} \tilde{\gamma}_\mu Y^c_{\xi,\mu} \right)_l ,$$

where $\tilde{\gamma}_\mu = (\tilde{\gamma}_{\mu,l})_{l \in \mathcal{L}}$ and $Y^c_{\xi,\mu}$ is the accumulated depreciation factor of consumption between the nodes ξ and μ (in other words, $Y^c_{\xi,\xi}$ is the identity matrix, and for every $\mu > \xi$, $Y^c_{\xi,\mu}$ is the product of the depreciation matrices Y^c_ν, where $\xi < \nu < \mu$).

We say that the price of good $l \in \mathcal{L}$ is free of bubbles, *in the weak sense*, at node ξ, if there is a deflator vector Γ_ξ such that $p_{\xi,l} = \mathrm{F}(\xi, \Gamma_\xi, l)$.

If for every possible deflator Γ_ξ the good l does not exhibit bubbles, then we say that the price $p_{\xi,l}$ is free of bubbles *in the strong sense*.

Using the deflators Γ_ξ and recursively applying equation (53) we have

$$\text{(61)} \quad p_{\xi,l} = \frac{1}{\gamma_\xi} \left(\sum_{\mu \in \mathcal{D}(\xi): \tilde{t}(\mu) < T} \tilde{\gamma}_\mu Y^c_{\xi,\mu} \right)_l + \frac{1}{\gamma_\xi} \left(\sum_{\mu \in \mathcal{D}(\xi): \tilde{t}(\mu) = T} \gamma_\mu p_\mu (Y^c_{\xi,\mu}) \right)_l , \quad \forall l \in \mathcal{L}, \ \forall T > \tilde{t}(\xi).$$

Denoting by $\mathrm{F}_T(\xi, \Gamma_\xi, l)$ the first term of the right side of equation (61), we have that the sequence $(\mathrm{F}_t(\xi, \Gamma_\xi, l))_{t > \tilde{t}(\xi)}$ is increasing and upper-bounded by the commodity price $p_{\xi,l}$ at ξ. Therefore the sequence converges and its limit is the fundamental value $\mathrm{F}(\xi, \Gamma_\xi, l)$.[15]

Hence the limit of the second term of the right side of (61) is well-defined, independently of the choice of deflator. Furthermore we have the following result:

Lemma 1: *Given an admissible prices vector (p, q, β), the price of a comodity $l \in \mathcal{L}$ has no bubbles at a node $\xi \in \mathcal{D}$ (in the weak sense) if and only if there is a vector of deflators Γ_ξ, for the successor nodes of ξ, such that*

$$\text{(62)} \qquad \lim_{T \to \infty} \left(\sum_{\mu \in \mathcal{D}(\xi): \tilde{t}(\mu) = T} \gamma_\mu p_\mu (Y^c_{\xi,\mu}) \right)_l = 0.$$

[15]This also proves that, given admissible prices, the fundamental value of a commodity l is always well-defined, independently of the choice of deflator Γ_ξ.

- **Assets in $\mathcal{J}^b$.**

The *fundamental value* at node ξ of an asset $j \in \mathcal{J}^b(\xi)$ is given by the discounted value, using the deflators Γ_ξ, of the dividends delivered at the successors of node ξ,

$$\text{(63)} \qquad \mathrm{F}(\xi, \Gamma_\xi, j) \equiv \frac{1}{\gamma_\xi}\left[\sum_{\mu \in \mathcal{R}(j):\mu > \xi} \gamma_\mu p_\mu A(\mu, j)\right].$$

We say that the price of an asset $j \in \mathcal{J}^b(\xi)$ is free of bubbles, *in the weak sense*, at node ξ, if there is a deflator vector Γ_ξ such that $q_{\xi,j} = \mathrm{F}(\xi, \Gamma_\xi, j)$. If for every possible deflator Γ_ξ the asset j does not exhibit bubbles, then we say that the price of j is free of bubbles *in the strong sense*.

The next result, which follows directly from the recursive application of equation (54) along the sub-tree $\mathcal{D}(\xi)$ and from arguments analogous to those made for commodities $l \in \mathcal{L}$, characterizes the existence of bubbles in the prices of assets in $\mathcal{J}^b$:

Lemma 2: *Given an admissible prices vector (p, q, β), the price of an asset $j \in \mathcal{J}^b(\xi)$ has no bubbles at $\xi \in \mathcal{D}$ (in the weak sense) if and only if there is a deflator vector Γ_ξ, for the successor nodes of ξ, such that the following transversality condition holds:*

$$\text{(64)} \qquad \lim_{T \to \infty} \sum_{\mu \in \mathcal{D}(\xi):\tilde{t}(\mu) = T} \gamma_\mu q_{\mu,j} = 0.$$

Moreover, if the asset $j \in \mathcal{J}^b$ is finite-lived (that is, there is $T \in \mathbb{N}$ such that $\tilde{t}(\mu) < T$, for every node $\mu \in \mathcal{R}(j)$), then there do not exist bubbles at ξ in the strong sense.

- **Asset in $\mathcal{J}^d$**

The *fundamental value of the purchase price*, at node ξ, of an asset $j \in \mathcal{J}^d(\xi)$, is given by the discounted value at node ξ of the dividends delivered by the asset to a lender (of one unit of the asset) who maintains his long position in the asset over time. Such dividends include both the value of the promises and the non-linearities that arise from the non-negativity condition on the long position at each successor node of ξ,

$$\text{(65)} \qquad \mathrm{F}_B(\xi, \Gamma_\xi, j) = \frac{1}{\gamma_\xi}\left[\sum_{\mu \in \mathcal{R}(j):\mu > \xi} \gamma_\mu L_{\mu,j} + \sum_{\mu \in \mathcal{R}(j):\mu \geq \xi}\left(\sum_{l \in \mathcal{L}} \gamma_{\mu,l} \tilde{\beta}_{\mu,j} (C^{1,L}_{\mu,j})_l + \gamma^L_{\mu,j}\right)\right],$$

Analogously, the *fundamental value of the sell price*, at node ξ, of an asset $j \in \mathcal{J}^d(\xi)$, is given by the discounted value at node ξ of the dividends received by a borrower (of one unit of the asset) who maintains his short position in the asset over time. Such dividends include, at each node μ, the difference between the depreciated value of the collateral requirements constituted at μ^- and the new margin requirements that must that must be constituted at μ: $\Delta\mathrm{CV}_{\mu,j} \equiv \mathrm{DCV}_{\mu,j} - \mathrm{CV}_{\mu,j}$. Thus,

$$(66) \qquad \mathrm{F}_S(\xi,\Gamma_\xi,j) = \frac{1}{\gamma_\xi}\left[\sum_{\mu\in\mathcal{R}(j):\mu>\xi}\gamma_\mu\left(\Delta\mathrm{CV}_{\mu,j} - B^h_{\mu,j}\right)\right] + \frac{1}{\gamma_\xi}\left[\sum_{\mu\in\mathcal{R}(j):\mu\geq\xi}\gamma^B_{\mu,j}\right] + \frac{1}{\gamma_\xi}\left[\sum_{\mu\in\mathcal{R}(j):\mu\geq\xi}\sum_{l\in\mathcal{L}}\gamma_{\mu,l}\tilde{\beta}_{\mu,j}\left(C^{1,B}_{\mu,j} + \sum_{j'\in\mathcal{J}^d(\mu)} C^{1,L}_{\mu,j'}\tilde{\beta}_{\mu,j'}(C^2_{\mu,j})_{j'}\right)_l\right],$$

Analogously to the case of commodities and assets in $\mathcal{J}^b$, we say that there are no bubbles (in the weak sense) in the purchase price (respectively, in the sell price) of asset $j \in \mathcal{J}^d$, at node ξ, if there is a deflator vector Γ_ξ such that $q_{\xi,j} = \mathrm{F}_B(\xi,\Gamma_\xi,j)$ (resp. $\mathrm{CV}_{\xi,j} - q_{\xi,j} = \mathrm{F}_S(\xi,\Gamma_\xi,j)$). The non-existence of bubbles in the strong sense is equivalent to the above equations holding for *every* deflator vector Γ_ξ.

As in the case of other asset types, the existence of bubbles in an asset $j \in \mathcal{J}^d(\xi)$ depends on whether a certain transversality condition holds.

The next result follows from equation (56) (via its repeated application along the successors to node ξ) and from the fact that, at each node $\mu \in \mathcal{R}(j) \cap \mathcal{D}(\xi)$, the sell price of asset j: $\mathrm{CV}_{\mu,j} - \tilde{\beta}_{\mu,j}q_{\mu,j}$ is positive, due to the absence of utility penalties for agents who default.

Lemma 3 *An asset $j \in \mathcal{J}^d(\xi)$ does not have bubbles (in the weak sense), at node ξ, if and only if there exists a vector of deflators Γ_ξ such that the discounted value, at ξ, of asset j's collateral requirements at period T goes to zero, as T goes to infinity. That is,*

$$(67) \qquad \lim_{T\to\infty}\sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T}\gamma_\mu\mathrm{CV}_{\mu,j} = 0.$$

The assets in $\mathcal{J}^d(\xi)$ which are finite-lived (or which go to default in finite time) do not have bubbles, in the strong sense, at the node ξ.[16]

[16]We say that an infinite-lived asset in $\mathcal{J}^d$ *goes to default in finite time* if there is an integer T such that $\beta_{\mu,j} = 0$ for every $\mu \in \mathcal{D}_t,\ t > T$.

We have thus characterized necessary and sufficient conditions for the existence of bubbles in assets negotiated at a node $\xi \in \mathcal{D}$.

The main result of this section is the following Theorem, which sums up the lemmas above:

Theorem 2: *Given admissible prices (p, q, β) then all finite-lived assets (and all assets which go to default in finite time) are free of bubbles, in the strong sense, at the nodes where they are traded.*

Moreover, a sufficient condition for the non-existence of bubbles, in the weak sense, for those assets that are traded at node ξ, is that the following two transversality conditions hold for a vector of deflators Γ_ξ,

$$\lim_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \gamma_\mu p_\mu = 0, \tag{68}$$

$$\lim_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \gamma_\mu q_{\mu,j} = 0, \quad \forall j \in \mathcal{J}^b(\xi). \tag{69}$$

The proof follows from the lemmas above and from Assumptions B and C; see the Appendix B.

Remark 2: It follows from the Theorem above that, in order to prevent the existence of bubbles in the assets which are negotiated at a node ξ, it is sufficient to guarantee the absence of speculation in the prices of those durable goods and assets which are not subject to default.

Within the context of the literature on infinite-horizon economies without default, Santos and Woodford (1997) and Magill and Quinzii (1996) have characterized the conditions that guarantee the non-existence of bubbles in asset prices. They show that if aggregated endowment of the economy is bounded by a portfolio trading plan and furthermore the asset has positive net supply, then the equilibrium prices exhibit no bubbles. If the markets are incomplete, then the occurrence of bubbles in equilibrium prices of assets with zero net supply might have real effects on the economy; that is, there might not be other bubble-free equilibrium prices which support the same allocations.

In our case, since there are assets which are subject to default, the markets might become more incomplete as time goes by (due to the closure of markets in assets which default), and therefore it might become more difficult to guarantee the existence of a finite-cost portfolio that bounds the aggregated endowment.

Hence one might expect that in our model the sufficiency conditions for the validity of equations (68) and (69) turn out to be more restrictive than those imposed by Santos and Woodford

(1997) or Magill and Quinzii (1996), and therefore that there might be in a sense more speculation in equilibrium.

5 Concluding Remarks

In this work we study an infinite-horizon sequential economy whose assets are either subject to default and protected by collateral or not subject to default but with restricted short sales. Such restrictions consist either of exogenous bounds on the amount that may be sold at each node, or of margin requirements which index the amount of short sales to the supply of goods and assets with bounded short-sales.

We study the existence of equilibrium and generalize the result of Araujo, Páscoa and Torres-Martínez (2000) to the case of multiperiod assets, incomplete participation of the agents, and financial collateral requirements. That is, if we assume that the only enforcement in case of default is the seizure of the collateral requirements by the lenders, we show that the collateral requirements (which might depend on prices but are taken as given by the agents) guarantee the existence of equilibrium without the need of a priori ad-hoc conditions that bound indefinite debt accumulation by the traders.

We also study speculation in the economy's admissible prices. We show that the existence of bubbles is caused by speculation either with durable goods or with default-proof assets. We therefore guarantee that the non-existence of bubbles at a given node is assured by the validity of transversality conditions on the prices of durable goods and of assets free of default.

It would be interesting to study restrictions which prevent bubbles in the default-proof assets or in other words, restrictions that guarantee that the aforementioned transversality conditions hold. Such restrictions (which we believe might be analogous to those imposed by Magill and Quinzii (1996) and Santos and Woodford (1997)) might have to guarantee that the economy's aggregate endowment is replicated with finite cost at each node of the tree.

On the other hand the fact that we allow the collateral requirements to depend on the commodity- and asset-prices, is in a certain sense, an endogeneization of such restrictions. It would nonetheless be interesting to allow the markets to have personalized collateral requirements which depend on the default level of each agent. We could in this way exclude the utility penalties, which are less natural than credit restrictions that are proportional to default levels.

In order to allow such personalized collateral requirements it might be necessary to allow a continuum of agents, which would avoid the non-convexities caused by the dependence of future collaterals on current renegotiation rules. All of these questions will be the subject of future research.

6 Appendix A

7 Appendix B

Proof of Theorem 2 Clearly the validity of condition (69) guarantees the non-existence of bubbles in the price of assets in $\mathcal{J}^b(\xi)$.

Now, it follows from condition (d.) of Theorem 1 that

$$0 < \lim_{t\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \gamma_\mu p_\mu Y^c_{\xi,\mu} \leq \lim_{t\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} (\kappa)^{T-\tilde{t}(\xi)} \gamma_\mu ||p_\mu||_\Sigma(1,1,\ldots,1). \tag{70}$$

It therefore follows from Lemma 1 (remember that $\kappa \in (0,1)$) that the validity of the transversality condition (68) guarantees the non-existence of bubbles in durable good prices.

Thus it is sufficient to prove that if an asset in $\mathcal{J}^d(\xi)$ exhibits bubbles then at least one of the conditions (68)-(69) does not hold.

If there is some asset $j \in \mathcal{J}^d(\xi)$ at node ξ with bubbles, then by Lemma 3 we have

$$\limsup_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \gamma_\mu \mathrm{CV}_{\mu,j} > 0. \tag{71}$$

Therefore Assumptions C and D imply that if $j \in \mathcal{A}_k$ then

$$\begin{aligned} 0 \quad < \quad & \sum_{l\in\mathcal{L}} \limsup_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \overline{C}_{j,l}\gamma_\mu p_{\mu,l} + \sum_{j'\in\mathcal{J}^b(\xi)\cup\mathcal{J}^m(\xi)} \limsup_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \overline{C}_{j,j'}\gamma_\mu q_{\mu,j'} \\ & + \sum_{j'\in\mathcal{A}_r\cup\mathcal{J}^d(\xi),\ r<k} \limsup_{T\to\infty} \sum_{\mu\in\mathcal{D}(\xi):\tilde{t}(\mu)=T} \overline{C}_{j,j'}\gamma_\mu q_{\mu,j'}. \end{aligned}$$

Hence, either one of the transversality conditions given by equations (68)-(69) does not hold, or there is an asset $j' \in \mathcal{A}_r$, where $r < k$, which exhibits bubbles at ξ.

Repeating the argument above for the assets in $\mathcal{A}_r \cap \mathcal{J}^d(\xi)$ it follows that the existence of bubbles in an asset $j \in \mathcal{J}^d(\xi)$ implies either that one of the transversality conditions given by the Theorem 2.2 does not hold, or that there is an asset $j' \in \mathcal{A}_0 \cap \mathcal{J}^d(\xi)$ which exhibits bubbles.

Since the collateral of the assets j' in $\mathcal{A}_0$ consists of consumption goods or of assets in $\mathcal{J}^b \cup \mathcal{J}^m$, the existence of bubbles in j' implies that one of the transversality conditions does not hold. This ends the proof. □

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