Estimation and accuracy of the stationary solution in the dynamic programming problem: New results

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29 de outubro de 2002

A CONTRACTIVE METHOD FOR COMPUTING THE STATIONARY SOLUTION OF THE EULER EQUATION*

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Abstract. A contractive method for computing stationary solutions of intertemporal equilibrium models is provided. The method is implemented using a contraction mapping derived from the first-order conditions. The deterministic dynamic programming problem is used to illustrate the method. Some numerical examples are performed.

1. Introduction

In intertemporal economic models one of the main difficulties is to find accurate estimatives of the stationary solutions. For instance, in dynamic programming models the traditional method is based on the Bellman approach. This consists in estimating the corresponding value function using a contraction mapping (see Stokey and Lucas with Prescott (1989)) and then computing the policy function from the approximated value function.

Taylor and Uhlig (1990) described some numerical methods based on the Bellman iterations (i.e., *Value-Function Grid* and *Quadrature Value-Function Grid* methods) and Santos and Vigo-Aguiar (1998) and Maldonado and Svaiter (2001) provided an estimation error for the approximate policy. However, Bellman's method has two main disadvantages: it is only useful when the model can be expressed as a representative agent model and, besides that, the speed of convergence is slow. On the other hand, numerical methods based on the

* This paper was presented at the 8th International Conference of the Society for Computational Economics in June, 2002. The authors gratefully acknowledge the comments of K. Judd. We also thank Caio Ibsen and David Moura for helping us with the numerical implementation.

† I would like to thank the CNPq of Brazil for the financial support 300059/990(RE).

‡ This research was supported by CNPq of Brazil.

solution of the Euler equation have been more efficient in the two aspects above: they can be used even when there is no representative agent and they are faster when an adequate approximation scheme is performed (see Judd (1998)).

The backward iteration algorithm, based on the Euler equation, provides a sequence of functions which converges pointwisely to the stationary solution. To implement this algorithm, Baxter *et.al* (1990) used a discretized version of the original problem. And using Santos and Vigo-Aguiar (2000) result, we know that the distance between the solution of the discretized problem and the solution of the original one is of order one in the grid mesh. Coleman (1990, 1991) used a similar approach, however the discretization is not necessary since linear interpolations are performed. Other numerical methods based on Euler equations are the projection method (Judd (1992)) and the parameterizing-expectations method (Marcet (1988) and Marcet and Lorenzoni (1998)).

The goal of this paper is also to provide a numerical method based on Euler equations to compute accurate estimatives of stationary solutions. For this, we define a contraction mapping which has the stationary solution as its fixed point. Moreover, the metric that we use is the C^1 uniform convergence. An important feature of our method is that it also provides an algorithm that compute the stationary solution for each state variable value. That is, we do not need to make a grid of the state variables for solving the Euler equation.

The convergence of our method requires a condition which is slightly stronger than the determinacy of the steady state condition. For the dynamic programming problem this amounts to the dominant diagonal condition. Some numerical examples are provided where this condition is satisfied for a realistic parameter set of the models.

The paper is divided as follows. In section 2 we present the deterministic model and its main hypotheses. In section 3 our contractive method based on the Euler-equation is presented. Finally, section 4 implements the algorithm derived from this method and gives some applications. All proofs are given in the appendix.

2. Basic Framework

The intertemporal equilibrium models that we are going to deal with are described by the following elements: $X \subset \Re^n$ is the state space, $D \subset X \times X \times X$ represents the intertemporal feasibility set and $E : D \to \Re^n$ is the function whose zeros define the temporary equilibria. We will assume that E is a twice continuously differentiable such that E_2 is negative definite on the interior of D.[1]

Let $|.|$ be one of the equivalent norms of $\Re^n$. The associated norm for the real square matrices space of order n ($\mathcal{M}_n$) will be $\|\,.\,\|$.[2] Finally, let $B_r(x) = \{y \in \Re^n\,;\, |y - x| < r\}$.

An equilibrium path from $x_0 \in X$ is a sequence $(x_t)_{t \geq 0}$ such that

$$E(x_{t-1}, x_t, x_{t+1}) = 0$$

and a stationary solution for (X, D, E) is a function $g\colon X \to X$ such that:

$$E(x, g(x), g^2(x)) = 0$$

[1] Since E is a function of $(x_1, x_2, x_3) \in D$, E_j is the vector of partial derivatives of E with respect to x_j.

[2] $\|X\| = \sup_{\{x \in \Re^n\,;\,|x|=1\}} |Xx|, \quad X \in \mathcal{M}_n$.

for all $x \in X$.

We say that $\bar{x} \in \Re^n$ is a *steady state* if $g(\bar{x}) = \bar{x}$. We will make the following:

Assumption D. *There exists an interior steady state* $\bar{x}$ *and* α *such that*[3]

(i) $\|(E_2)^{-1}E_1\| + \|(E_2)^{-1}E_3\| < \alpha < 1$

(ii) $\|(E_2)^{-1}E_3\| < 1/2$

Remark: The condition for the existence of a locally unique stationary equilibrium is that the steady state must be a saddle point of the linearization of $E = 0$. A necessary and sufficient condition for this is:[4]

$$\|(E_2)^{-1}E_1 + (E_2)^{-1}E_3\| < 1$$

which is weaker than assumption **D**.

A basic example of this structure is the dynamic programming problem. Following the notation of Stokey and Lucas with Prescott (1989),

$$E(x_{t-1}, x_t, x_{t+1}) = F_2(x_{t-1}, x_t) + \beta F_1(x_t, x_{t+1})$$

where F is the return function, β is the discount factor and the set D is defined from technological constraints. In this case g represents the policy function. It is easy to verify that for the one-dimensional case, assumption **D (i)** amounts to the dominant diagonal condition of the Jacobian of E at the steady state.

3. Main Result

In this section we will provide an iterative method for computing stationary solutions in a neighborhood of a steady state for the model (X, D, E). This method consists in defining an implicit map from the temporary equilibrium equation and showing that this map is a contraction with the stationary solution as the fixed point.

Given $r > 0$, $\gamma > 0$ and $\bar{x} \in \Re^n$ let us denote

$$\mathcal{H} = \{h \in C^2(B_r(\bar{x}))\,;\, h(\bar{x}) = \bar{x},\ \|Dh(x)\| \leq \alpha \text{ and } \|D^2h(x)\| \leq \gamma,\ \forall x \in B_r(\bar{x})\}$$

where $C^l(B_r(\bar{x}))$ is the space of l-th continuously differentiable functions from $B_r(\bar{x})$ into itself and γ is a constant.

Define the norm

$$\|h\|_1 = \sup_{x \in B_r(\bar{x})} \|Dh(x)\|\ ,\ \text{for all } h \in \mathcal{H}.$$

Let $\bar{\mathcal{H}}$ be the closure of $\mathcal{H}$ with respect to this norm. Therefore, $(\bar{\mathcal{H}}, \|\cdot\|_1)$ is a complete metric space. Observe that, by the definition of the metric (uniform convergence in the first derivative), it is easy to see that $\bar{\mathcal{H}}$ is a subset of $C^1(B_r(\bar{x}))$.

[3] The derivatives are evaluated at $(\bar{x}, \bar{x}, \bar{x})$. Observe that (i) implies (ii) when $n = 1$.

[4] The quadratic equation $x^2 + ax + b = 0$ has a root with modulus greater than one and the other lower than one if and only if $|a| > |1 + b|$.

Lemma 3.1. *Under* **D**, *there exist* $r > 0$, $\gamma > 0$ *and* $\varphi : \mathcal{H} \to \mathcal{H}$ *such that*

$$E(x, \varphi_h(x), h^2(x)) = 0 \qquad (*)$$

for all $x \in B_r(\bar{x})$ *and* $h \in \mathcal{H}$.

Theorem 3.2. *Assume* **D**. *Then there exist* $r > 0$ *and* $\gamma > 0$ *such that* $\varphi: \mathcal{H} \to \mathcal{H}$ *defined in Lemma 3.1 is a* η*-contraction, for some* $\eta \in (0, 1)$.

The proof of Theorem 3.2 shows that the map $\varphi: \bar{\mathcal{H}} \to \bar{\mathcal{H}}$ is a $\eta-$contraction and consequently has a fixed point. It is easy to see that such a fixed point is a stationary solution of (X, D, E). Therefore, the map φ provides a recursive method for computing this solution in a neighborhood of the steady state.

The following corollary shows that the recursive method obtained from Theorem 3.2 holds in every neighborhood where assumption **D** is satisfied.

Corollary 3.3. *If the assumption* **D** *is satisfied in a convex neighborhood* $\mathcal{N}$ *of* $\bar{x}$, *then there exist* $\gamma > 0$ *and* $\varphi: \mathcal{H} \to \mathcal{H}$ *satisfying* $(*)$ *(where* $B_r(\bar{x})$ *is replaced by* $\mathcal{N}$*) and there exist* $N \geq 1$ *and* $\eta < 1$ *such that* φ^N *is a* η*-contraction with fixed point* g.

4. The Algorithm and Numerical Examples

In this section we will describe the algorithm derived from Theorem 3.2. But first we will discuss other methods in the literature.

Let us consider the dynamic programming problem. The first method consists in defining the following operator from the Euler equation: given a feasible map $h : X \to X$, let us define $Th : X \to X$ implicitly by

$$F_2(x, Th(x)) + \beta F_1(Th(x), h(Th(x))) = 0.$$

The optimal policy function g is a fixed point of T and the sequence $(T^n(h_0))_n$ converges to g pointwise, where h_0 is constant.[5] Indeed, we claim that this approach is equivalent to Bellman's method. To see this, define the following sequence of functions:

$$v_n(x) = \max_{\{y;\,(x,y)\in A\}} F(x, y) + \beta v_{n-1}(y)$$

for all $n \geq 1$, where A is the feasible set related with the problem and $v_0(x) = F(x, h_0(x))$. From the first order condition and the Envelope Theorem:

$$F_2(x, h_n(x)) + \beta F_1(h_n(x), h_{n-1}(h_n(x))) = 0,$$

[5] A constant function may not be feasible. In this case we have to choose, for instance, a piecewise constant h_0.

where $h_n(x) = \underset{\{y;\,(x,y)\in A\}}{\operatorname{argmax}} F(x,y) + \beta v_{n-1}(y)$. Using the definition of T and this last equation, it is easy to see that $h_n = T^n(h_0)$. Hence, the Bellman method implies that h_n converges to g. In particular, this method is equivalent to Bellman's one.

Baxter *et al.* (1990)) implements that method making a discretization of the state space, whereas Coleman (1990, 1991) used a linear interpolation in each step.

Li (1998) uses a similar method to ours: she defines the same mapping φ of Theorem 3.2 and proves that it is a contraction in the C^0 topology. In Example 2 below we show that our approach has the following advantages: (i) the set of economies where the contractive property holds is larger than hers; (ii) the convergence is in the C^1 topology and, in particular, the stationary solution is C^1.

The Algorithm

We now describe the main steps of the algorithm which implements our contractive method.[6] The main difference of our implementation with respect to the methods above is that it is not necessary to make a discretization of the state space. More precisely, we can give an accurate approximation of the stationary solution for each $x \in X$.

Let $h_0 : X \to X$ be a constant function (for instance, $h_0 \equiv \bar{x}$, the steady state). Fix $x \in X$.

computing h_1: solve

$$E(x, y, h_0^2(x)) = 0.$$

computing $h_2(x)$

first step: find $h_1^2(x)$, i.e., solve

$$E(h_1(x), y, h_0^2(h_1(x))) = 0.$$

second step: solve

$$E(x, y, h_1^2(x)) = 0$$

for finding $h_2(x)$.

In general, we have to proceed as follows:

computing $h_{n+1}(x)$

first step: find h_n^2. In order to do this, we have to compute, using the equilibrium equation, the following sequence

$$h_1 h_n,\ h_1^2 h_n,\ h_2 h_n,\ h_1 h_2 h_n,\ h_1^2 h_2 h_n,\ h_2^2 h_n,\ h_3 h_n, ..., h_{n-1}^2 h_n,\ h_n^2$$

where x was dropped in the notation.

[6]For the examples, we used MATLAB to implement the numerical routines. Upon request, we will provide the MATLAB code by E-mail. The E-mail address is humberto@fgv.br.

second step: solve

$$E(x, y, h_n^2(x)) = 0$$

for finding $h_{n+1}(x)$.

Therefore, as we can see the algorithm can compute $h_n(x)$ from solely $h_{n-1}(x)$ for each $x \in X$ without making any discretization of the state space in previous steps.

The Examples

1. Deterministic Growth Model

Consider the classical deterministic growth model with utility and production functions:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \text{ and } f(k) = Ak^{\alpha}$$

where $\gamma \geq 0$ (for $\gamma = 1$, $u(c) = \ln c$) and $0 < \alpha \leq 1$.
The Euler equation is given by:

$$E(x, y, z) = -(x^{\alpha} - y)^{-\gamma} + \beta\alpha y^{\alpha-1}(y^{\alpha} - z)^{-\gamma} = 0.$$

For $\alpha = 1$ and $\gamma = 1$ it is possible to determine the operator φ explicitly

$$\varphi_h(k) = \frac{h^2(k) + \beta k}{1+\beta}$$

and the optimal policy function $g(k) = \beta k^{\alpha}$ which is a fixed point of φ. It is important to note that these explicit calculations are not possible for the remaining cases below and, therefore, this justifies the use of the proposed algorithm.

Using the Mean Value Theorem, we have the following estimative:

$$\|\varphi_{h_1} - \varphi_{h_2}\| \leq \frac{1 + \|Dh_1\|}{1+\beta}\|h_1 - h_2\|,$$

where $\|\cdot\|$ is the sup norm. Taking the domain of φ as $h_1 \in C^1$ such that $\|Dh\| \leq M < \beta$ it results that φ is a contraction. It is important to note our theorem guarantees a C^1-contraction probably in a smaller domain.

In figures 1 and 2 we show the numerical results of our method in two particular specification of the parameters: $\beta = .95$ and domain $[.05; 1]$. The initial function is $h_0(k) = \min\{\bar{k}, k\}$.

In figure 1, $u(c) = \ln(c)$, $\alpha = .34$ and the optimal policy function is the continuous line and the iterations h_1 and h_10 (which is close to g) are the dotted lines (the distance between h_9 and h_{10} is 1.349×10^{-7}). In figure 2, if $\gamma = .2$ and $\alpha = .5$, then the true optimal policy is not explicitly known and we only show the iterations h_1 through h_9 in dotted line and h_{10} in continuous line (the distance between h_{10} and g is 1.087×10^{-9}).

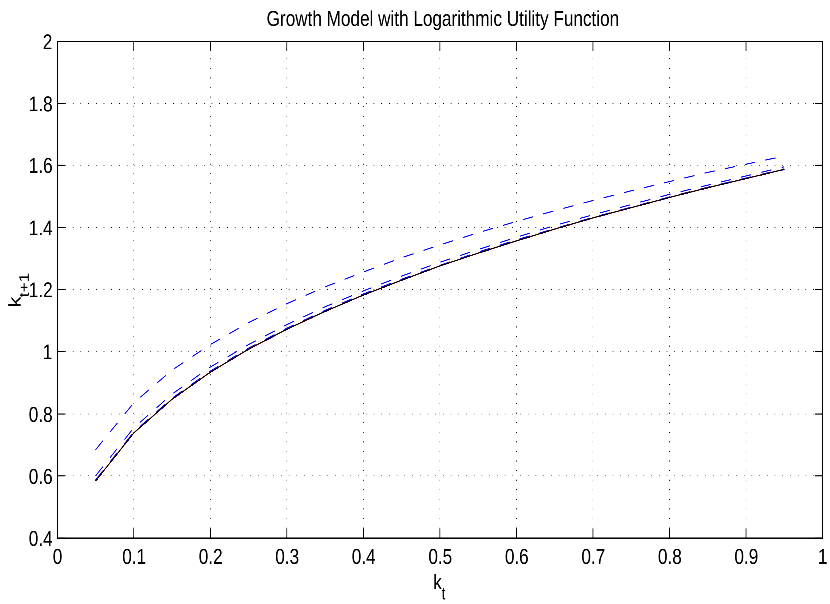

FIGURE 1: Deterministic growth model with $u(c) = \ln c$, $A = 5$, $\alpha = .34$, $\beta = .95$ and iterations$= 10$.

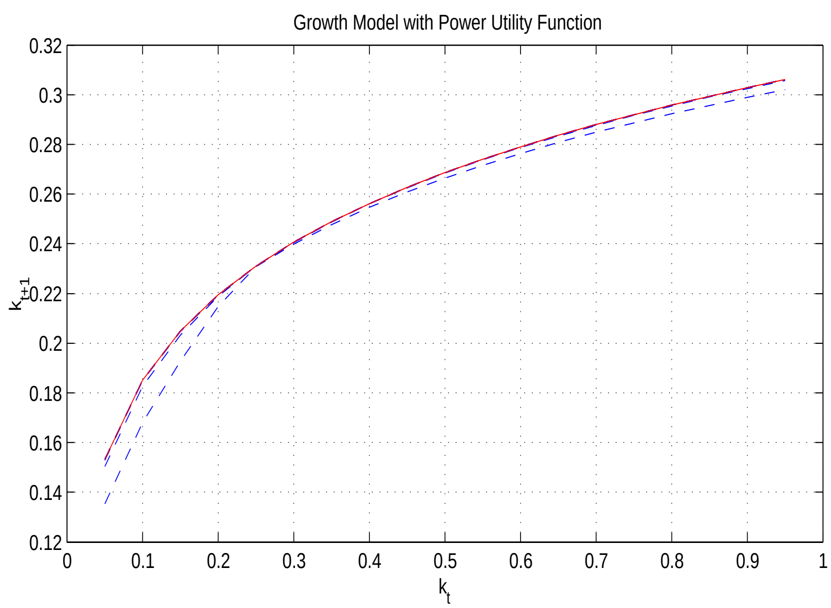

FIGURE 2: Deterministic growth model with $\gamma = .2$, $A = 5$, $\alpha = .5$, $\beta = .95$ and iterations$= 10$.

2. A Monetary Model (Li (1998))

Li (1998) presents a monetary model with preferences given by $U(c) = c^{1-A}/(1-A)$ ($A \geq 0$ and $A \neq 1$) and discount factor $0 < \beta < 1$. The intertemporal equilibrium equation for this model is given by:

$$E(p_{t-1}, p_t, p_{t+1}) = p_t + \phi\left(\frac{p_t}{p_{t+1}}\right)p_{t-1} - 1 = 0$$

where p_t is the (scaled) price of the economy in period t and $\phi(x) = \frac{\beta^{1/A}}{\beta^{1/A}+x^{(A-1)/A}}$.

A stationary solution is defined by a function $g : [0,1] \rightarrow [0,1]$ such that

$$1 - x\phi\left(\frac{g(x)}{g \circ g(x)}\right) = g(x),$$

which gives a steady state equilibrium value: $\bar{p} = 1/(1+\phi(1))$.

Our assumption **D** imposes the following bounds for the parameter values:

$$\begin{cases} 0 < \phi(1) < 1 - 2|\phi'(1)| & \text{if } \phi'(1) \in (-1, 0) \\ 0 < \phi(1) < 1 & \text{if } \phi'(1) \geq 0. \end{cases}$$

Thus, the required parameter value set in Li (1998) are strictly included in ours, as one can easely check. Moreover, the convergence is in the C^1-topology (instead of the C^0-convergence found by Li (1998)).

Figure 3 shows stationary solution approximations for the following parameter values: $A = .4$ and $\beta = .99$ and the interval of price is $[.05; 1]$. We made ten interactions where the continuous line represents the tenth one (the distance from the ninth is 1.361×10^{-8}).

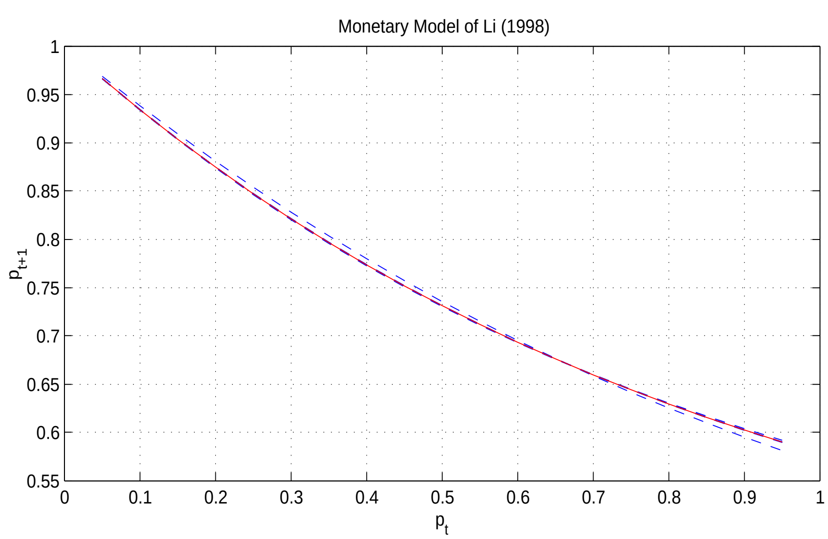

FIGURE 3: Stationary solution of the monetary model of Li (1998) with $A = .4$, $\beta = .99$ and iterations= 10.

3. Growth with Externalities Model (Boldrin and Rustichini (1994))

Boldrin and Rustichini (1994) analyzes a two-sector growth model with labor externality. The utility function is linear and the technological frontier is characterized by:

$$T(x, x', k) = (k^\eta - x')^\alpha (x - \gamma x')^{1-\alpha}$$

where x is the current capital value, x' is the next period capital value, k is the current aggregate capital stock which is considered as an externality in the total number of units of labor and $\alpha, \gamma \in (0, 1)$. In this case the intertemporal equilibrium equation is given by:

$$E(x_t, x_{t+1}, x_{t+2}) = T_2(x_t, x_{t+1}, x_t) + \beta T_1(x_{t+1}, x_{t+2}, x_{t+1}) = 0$$

and the unique interior steady state is

$$\bar{x} = \left(\frac{(\beta - \gamma)(1 - \alpha)}{(\beta - \gamma)(1 - \alpha) + (1 - \gamma)\alpha}\right)^{1/(1-\eta)}.$$

In this model, condition D is:

$$|T_{22} + \beta(T_{11} + T_{13})| > \beta|T_{12}| + |T_{12} + T_{23}|$$

which is stronger than the condition for the existence of a saddle steady state (see the remark after assumption **D**).

Figure 4 shows the approximations of the stationary solution around the steady state for parameter values that satisfy assumption **D**. The parameters are: $\alpha = .5$, $\beta = .95$, $\gamma = .5$, $\eta = .5$ and the interval of capital is $[.05; 1]$. We did ten interactions where the continuous line represents the tenth one (the distance from the ninth is 2.233×10^{-7}).

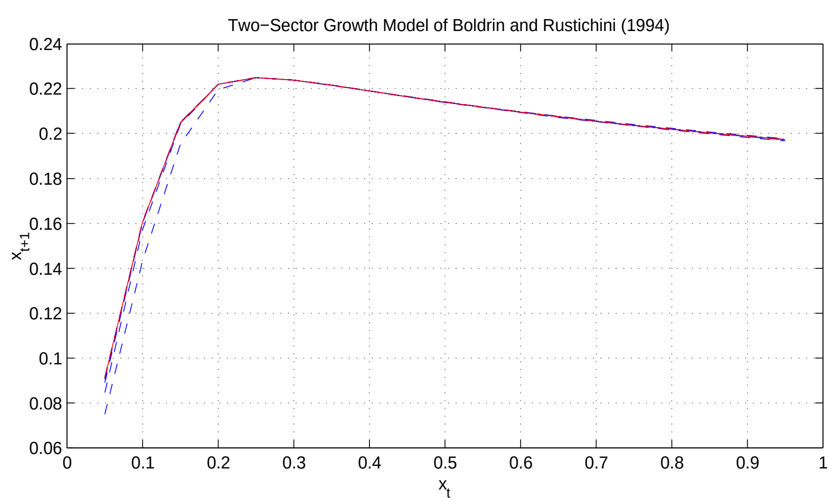

FIGURE 4: Stationary solution of the two-sector growth model of Boldrin and Rustichini (1994) with $\alpha = \eta = \gamma = .5$, $\beta = .95$ and iterations$= 10$.

5. Conclusions

In this paper we provide a recursive method to approximate the stationary solution of the Euler equation for intertemporal deterministic models. Its major difference from classical methods is that it is performed from a contraction mapping in the C^1-topology of a suitable functional space. In particular this implies the continuously differentiability of the stationary solution.

The required hypothesis is an open condition related with the first derivatives of the structural equations evaluated at the steady state. This hypothesis is slightly stronger than the determinacy of the steady state condition.

Another interesting feature of this method is that the algorithm that implements it needs neither a discretization of the state space nor a piecewise linear approximations of the iterations. Our algorithm is illustrated by numerical examples applied to some models in the literature.

APPENDIX

Proof of Lemma 3.1: Assumption **D** implies that there exists $r > 0$ such that $B_r(\bar{x}, \bar{x}, \bar{x}) \subset D$ and

$$\sup_{B_r(\bar{x},\bar{x},\bar{x})} \|E_2^{-1}E_1\| + \|E_2^{-1}E_3\| < \alpha \qquad (I)$$

and

$$\sup_{B_r(\bar{x},\bar{x},\bar{x})} \|E_2^{-1}E_3\| < 1/2. \qquad (II)$$

Let $h \in \mathcal{H}$ and observe that the function $(x_1, x_2) \mapsto E(x_1, x_2, h^2(x_1))$ defined on a neighborhood of $(\bar{x}, \bar{x})$ is twice continuously differentiable,

$$E(\bar{x}, \bar{x}, h^2(\bar{x})) = E(\bar{x}, \bar{x}, \bar{x}) = 0$$

and $E_2(x_1, x_2, h^2(x_1))$ is a negative definite matrix for all (x_1, x_2) in a neighborhood of $(\bar{x}, \bar{x})$. By the Implicit Function Theorem, there exist $\epsilon_1, \epsilon_2 \in (0, r)$ and a continuously differentiable function $\varphi : B_{\epsilon_1}(\bar{x}) \to B_{\epsilon_2}(\bar{x})$ $\quad (\varphi = \varphi_h)$ such that

$$E(x_1, x_2, h^2(x_1)) = 0 \ , \ x_i \in B_{\epsilon_i}(\bar{x}) \ , \ i = 1, 2 \iff x_2 = \varphi(x_1).$$

Moreover, for each x in $B_{\epsilon_1}(\bar{x})$,

$$D\varphi(x) = -E_2^{-1}(x, \varphi(x), h^2(x))[E_1(x, \varphi(x), h^2(x)) + E_3(x, \varphi(x), h^2(x))Dh(h(x))Dh(x)]$$

$$\Rightarrow \|D\varphi(x)\| \leq \sup_{B_r(\bar{x},\bar{x},\bar{x})} \|E_2^{-1}E_1\| + \|E_2^{-1}E_3\| < \alpha \ , \text{ for all } x \in B_{\epsilon_1}(\bar{x})$$

because of (I) and the fact that $\|Dh(x)\| \leq \alpha < 1$ on $B_r(\bar{x})$.

Observe that

$$\varphi(x) = \bar{x} + \int_0^1 D\varphi(\bar{x} + t(x - \bar{x}))(x - \bar{x})dt \ , \text{ for all } x \in B_{\epsilon_1}(\bar{x})$$

$$\Rightarrow |\varphi(x) - \bar{x}| \leq \sup_{t \in [0,1]} \|D\varphi(\bar{x} + t(x - \bar{x}))\| \, |x - \bar{x}| < \epsilon_1$$

and, therefore, we can suppose that $\epsilon = \epsilon_1 = \epsilon_2$.

We claim that we can take $\epsilon = r$. Let $r^* = \sup \{\epsilon > 0;\ \varphi \text{ is defined on } B_\epsilon(\bar{x})\}$. Suppose that $r^* < r$. First, for each $x_1 \in B_{r^*}(\bar{x})$, there exists a unique $x_2 \in B_{r^*}(\bar{x})$ such that $E(x_1, x_2, h^2(x_1)) = 0$. Otherwise, there exist x_2 and $\tilde{x_2}$, $x_2 \neq \tilde{x}_2$, satisfying the last equality. Define $f\colon [0,1] \to \Re$ by

$$f(t) = (x_2 - \tilde{x_2})' E(x_1, x_2 + t(x_2 - \tilde{x_2}), h^2(x_1)).$$

Then $f(0) = f(1) = 0$ and $f'(t) = (\tilde{x_2} - x_2)' E_2 (x_2 - \tilde{x_2}) < 0$. Since E_2 (calculated at $(x_1, x_2 + t(x_2 - \tilde{x_2}), h^2(x_1)))$ is a negative definite matrix, this is a contradiction.

Let x be a point on the border of $B_{r^*}(\bar{x})$. Let $(x_n)_{n \geq 0}$ be a sequence in $B_{r^*}(\bar{x})$ converging to x. Since $(\varphi(x_n))_{n \geq 0}$ is a sequence in $\overline{B_{r^*}}(\bar{x})$ (a compact set), there exists a subsequence converging to a point $y \in B_{r^*}(\bar{x})$. By the continuity of $E(\cdot, \cdot, h^2(\cdot))$, $E(x, y, h^2(x)) = 0$. However, y is uniquely determinated, implying that the sequence $(\varphi(x_n))_{n \geq 0}$ converges to y. We can apply again the Implicit Function Theorem at $(x, y, h^2(x))$ for the Euler equation. Doing this for all points on the border of $B_{r^*}(\bar{x})$, we can extend φ to a ball centered at $\bar{x}$ which contains (strictly) $B_{r^*}(\bar{x})$. This contradicts the definition of r^*.

Now we choose the constant $\gamma > 0$. Let us calculate the second order derivative of φ:

$$\begin{aligned} D^2\varphi(x) = &- [D(E_2^{-1}E_1) + D(E_2^{-1}E_3) Dh(h(x)) Dh(x)] \\ &- E_2^{-1}E_3 [D^2h(h(x))(Dh(x))^2 + Dh(h(x)) D^2h(x)] \end{aligned}$$

(calculated at $(x, \varphi(x), h^2(x))$. Taking the supremum on the right side and using (II) we have[7]

$$\|D^2\varphi(x)\| \leq c_1 + (\alpha^2\gamma + \alpha\gamma) \sup_{B_r(\bar{x},\bar{x},\bar{x})} \|E_2^{-1}E_3\| \leq c_1 + c_2\gamma$$

where $c_1 > 0$ and $0 < c_2 < 1$ (by (II)). Choose γ sufficiently large such that $c_1 + c_2\gamma \leq \gamma$.

Proof of Theorem 3.2: Let $\mathcal{B} \subset \mathcal{M}_n$ be the unit ball. Since E is C^2, there exists $r > 0$ such that the map

$$\Psi : B_r(\bar{x}, \bar{x}, \bar{x}) \times \mathcal{B}^2 \to \mathcal{M}_n$$

defined by

$$\Psi(x_1, x_2, x_3, M_1, M_2) = -E_2^{-1}(x_1, x_2, x_3)[E_1(x_1, x_2, x_3) + E_3(x_1, x_2, x_3) M_1 M_2]$$

is a Lipschitz function, i.e., there exists $L > 0$ such that:

$$\|\Psi(x^1, M^1) - \Psi(x^2, M^2)\| \leq L \sum_{i=1}^{3} |x_i^1 - x_i^2| + S_r[\|M_1^1 - M_1^2\| + \|M_2^1 - M_2^2\|] \qquad (III)$$

[7] Observe that one of the components of the derivative of $E_2^{-1}E_1$ and $E_2^{-1}E_3$ involves the derivatives of φ and h which are uniformly bounded by α.

where we are denoting $x^i = (x_1^i, x_2^i, x_3^i) \in B_r(\bar{x})^3$, $M^i = (M_1^i, M_2^i) \in \mathcal{B}^2$, $i = 1, 2$ and $S_r = \sup_{B_r(\bar{x},\bar{x},\bar{x})} \|E_2^{-1} E_3\|$.

The fact that E is C^1 and assumption **D** (ii) guarantee that we can choose $r > 0$ such that $S_r < 1/2$. Finally, we choose such r satisfying Lemma 3.1 and:

$$\frac{(Lr\alpha + S_r)(2 + \gamma r)}{1 - Lr} = \eta_r < 1$$

(notice that $|\eta_r - 2S_r| \to 0$ when $r \to 0$).

We will prove that the function $\varphi : \mathcal{H} \to \mathcal{H}$ defined by $\varphi(h) = \varphi_h$ is a η_r-contraction mapping (φ_h defined in Lemma 3.1).

Given $h_1, h_2 \in \mathcal{H}$, by Lemma 3.1, for $i = 1, 2$ and $x \in B_r(\bar{x})$,

$$D\varphi_{h_i}(x) = \Psi(x, \varphi_{h_i}(x), h_i^2(x), Dh_i(h_i(x)), Dh_i(x))$$

Observe that

$$Dh_i^2 = Dh_i(h_i)Dh_i, \; i = 1, 2$$

and

$$\begin{aligned} Dh_1(h_1)Dh_1 - Dh_2(h_2)Dh_2 =& (Dh_1(h_1) - Dh_2(h_1))Dh_1 \\ &+ (Dh_2(h_1) - Dh_2(h_2))Dh_1 + Dh_2(h_2)(Dh_1 - Dh_2). \end{aligned}$$

Thus,

$$\|h_1^2 - h_2^2\|_1 \leq \alpha\|h_1 - h_2\|_1 + \alpha\gamma r\|h_1 - h_2\|_1 + \alpha\|h_1 - h_2\|_1 \tag{IV}$$

by the definition of $\|\cdot\|_1$ and the space $\mathcal{H}$.

Therefore, from (III) and (IV)

$$\|\varphi_{h_1} - \varphi_{h_2}\|_1 \leq L(r\|\varphi_{h_1} - \varphi_{h_2}\|_1 + r\alpha(2 + \gamma r)\|h_1 - h_2\|_1) + (2 + \gamma r)S_r\|h_1 - h_2\|_1$$

$$\Rightarrow \|\varphi_{h_1} - \varphi_{h_2}\|_1 \leq \eta_r\|h_1 - h_2\|_1.$$

So $\varphi : \mathcal{H} \to \mathcal{H}$ is a η_r-contraction. It is easy to see that we can extend continuously φ to $\bar{\mathcal{H}}$. Let φ also denote this extension, then we obtain a η_r-contraction on $\bar{D}_r$. By the Banach Fixed Point Theorem, there exists $g \in \bar{D}_r$ such that $\varphi_g = g$ and

$$\|\varphi^n(h) - g\|_1 \leq (\eta_r)^n\|h - g\|_1, \quad \text{for all } h \in D_r.$$

Proof of Corollary 3.3: Fix $r > 0$ and let $\rho > 0$ be such that $\alpha(r + \rho) \leq r$ (*i.e.* $\rho \leq \frac{1-\alpha}{\alpha}r$) and assumption **D** holds in $B_{r+\rho}(\bar{x}) \cap C$. From assumption **D**$(i)$, the choice of ρ guarantees that for each $h \in \bar{D}_{r+\rho}$, $h(x) \in B_r(\bar{x})$ for all $x \in B_{r+\rho}(\bar{x})$. Let $B = B_r(\bar{x})$, $A = B_{r+\rho}(\bar{x}) - B$ and $|.|_i^A$, $|.|_i^B$ the C^i-norm for $i = 0, 1$ on A and B, respectively. For $h_1,\, h_2 \in D_{r+\rho}(\bar{x})$ we have:

$$|\varphi_{h_1} - \varphi_{h_2}|_1^A \leq L\{\rho|\varphi_{h_1} - \varphi_{h_2}|_1^A + |\varphi_{h_1} - \varphi_{h_2}|_0^B + \rho|h_1^2 - h_2^2|_1^A + |h_1^2 - h_2^2|_0^B\}$$

$$+S_{r+\rho}[\gamma|h_1 - h_2|_0^A + |h_1 - h_2|_1^B] + S_{r+\rho}\gamma|h_1 - h_2|_1^A$$

From definitions of $|.|_i^A$ and $|.|_i^B$ we have the following inequalities:

$$|h_1^2 - h_2^2|_1^A \leq \alpha|h_1 - h_2|_1^B + \alpha\gamma[\rho|h_1 - h_2|_1^A + |h_1 - h_2|_0^B] + \alpha|h_1 - h_2|_1^A;$$

$$|h_1^2 - h_2^2|_0^B \leq r(\alpha + 1)|h_1 - h_2|_1^B\,;\ |h_1 - h_2|_0^A \leq |h_1 - h_2|_0^B + \rho|h_1 - h_2|_1^A;$$

$$|\varphi_{h_1} - \varphi_{h_2}|_0^B \leq r|\varphi_{h_1} - \varphi_{h_2}|_1^B \leq r\eta_r|h_1 - h_2|_1^B.$$

Using these last three inequalities in the first one we get:

$$(1 - L\rho)|\varphi_{h_1} - \varphi_{h_2}|_1^A \leq [L\alpha\rho + S_{r+\rho}](1 + \gamma\rho)|h_1 - h_2|_1^A$$

$$+[L\alpha\rho(1 + \gamma) + Lr(\eta_r + \alpha + 1) + S_{r+\rho}(r\gamma + 1)]|h_1 - h_2|_1^B.$$

Defining

$$\begin{cases} \alpha_\rho = [L\alpha\rho + S_{r+\rho}](1 + \gamma\rho)(1 - L\rho)^{-1} \\ \gamma_\rho = [L\alpha\rho(1 + \gamma) + Lr(\eta_r + \alpha + 1) + S_{r+\rho}(r\gamma + 1)](1 - L\rho)^{-1} \end{cases}$$

we have that

$$|\varphi_{h_1} - \varphi_{h_2}|_1^A \leq \alpha_\rho|h_1 - h_2|_1^A + \gamma_\rho|h_1 - h_2|_1^B.$$

By induction

$$|\varphi_{h_1} - \varphi_{h_2}|_1^A \leq \alpha_\rho^n|h_1 - h_2|_1^A + \gamma_\rho\Big(\sum_{i=0}^{n-1} \alpha_\rho^{n-1-i}\eta_\rho^i\Big)|h_1 - h_2|_1^B \leq \gamma_n|h_1 - h_2|_1^{A\cup B}.$$

where

$$\gamma_n = \text{Max}\{\alpha_\rho^n\,,\ n\gamma_\rho(\text{Max}\{\alpha_\rho\,,\ \eta_\rho\})^{n-1}\}.$$

Since $\alpha_\rho \to S_r < 1/2$ when $\rho \to 0$ and $\eta_r < 1$ then, for all $\rho > 0$ small enough, we can find $n > 1$ such that $\gamma_n < 1$, so φ^n is a γ_n-contraction.

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On the accuracy of the estimated policy function using the Bellman equation

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September 17, 2002

Abstract

In this paper we give explicit error bounds for approximations of the optimal policy function in the stochastic dynamic programming problem. The approximated policy function is obtained by using the Bellman equation with an approximated value function and the error bounds depend on the primitive data of the problem. Neither differentiability of the return function nor interiority of solutions is required. Furthermore, similar error bounds are obtained when the maximization in the Bellman equation and the computation of the associated policy function are performed inexactly. This shows the robustness of the method and provides a stopping criterium for computational implementations.

Keywords: Stochastic dynamic programing problem, estimation of the policy function

JEL Classification Numbers: C61, D90

*I would like to thank the CNPq of Brazil for financial support 300059/99-0(RE).

†Partially supported by CNPq Grant 301200/93-9(RN) and by PRONEX–Optimization. Finally, we would like to thank the valuable comments and suggestions of Kenneth Judd.

1 Introduction

It is well known that the Bellman Principle of Optimality allows to define an iterative method to estimate the value function and the optimal policy function in stochastic dynamic programming problems (Christiano [4], Tauchen [11], Santos and Vigo-Aguiar [9]). This method is based on a contraction mapping and provides an efficient algorithm to estimate the value function with high precision. However, until now, only asymptotic convergence results, without numerical error bounds, were obtained for the estimated policy function with this method. For example, Stokey and Lucas with Prescott ([10], theorem 9.9) established the pointwise convergence (and uniform convergence if the domain is a compact set) to the optimal policy function under the assumption of strict concavity of the return function.

We will prove that the error of the estimated policy function in the n-th step of the contractive method is bounded by

$$\left[\frac{2}{\eta_1}\left(\frac{\|F\|_\infty}{1-\beta}\right)\right]^{1/2}\beta^{n/2}.$$

where F is the return function, β is the discount factor, η_1 is the parameter of strong concavity of F and n is the number of iterations. This estimate holds for a constant function equal to zero as an initial guess for the value function.

There exist other approaches for obtaining good estimates for the optimal policy function. For example, the Euler equation grid method (Baxter *et al.* [1], Coleman [2, 3]), the parameterized expectations method (Marcet and Marshall [6]) and projection methods (Judd [5]). Again, only asymptotic convergence to the optimal policy function was proved for these methods.

Bounds for the distance between the optimal policy function (of the original problem) and the *exact* optimal policy function of a discretized (piecewise linear) version of the problem were obtained by Santos and Vigo-Aguiar [9].

In Santos [8], Euler equation residuals were used to obtain error bounds for an approximated policy function. Either an assumption on the repeated iterations of the approximated policy function (condition NDIV) or a bound on the second derivative of the return function evaluated at the optimal policy function was necessary. Existence of interior solutions and twice differentiablity of the return function were also required.

The error bounds presented in this paper only require boundedness and strong concavity of the return function. These assumptions are quite general and were used by Santos and Vigo-Aguiar [9] and Santos [8]. We require neither differentiability of the return function nor existence of interior solutions.

We also prove the robustness of the contractive method by considering the use of an inexact operator defining the contractive method or inexact solutions in each maximization process. In this case we again obtain error bounds.

Our result has a direct consequence for the use of the contractive method to estimate the policy function: the number of iterations required to attain a given precision is computed in advance, using only some primitive data of the problem.

This paper is organized into five sections. Section 2 describes the framework we will consider and the hypotheses assumed. Section 3 states the main theorem, which provides an error bound for the estimated policy function when the contraction method based on Bellman's principle is used. Section 4 shows the robustness of this method under small numerical errors. Conclusions are given in Section 5 and the proofs are given in the appendix.

2 The framework

The stochastic dynamic programming problem is defined using the following elements: the set of values for the endogenous state variables $X \subset \mathbb{R}^l$ (which is a convex Borel set), the set of values for the exogenous shocks $Z \subset \mathbb{R}^k$ (which is a compact set); both are measurable spaces with their σ-algebras denoted by $\mathcal{X}$ and $\mathcal{Z}$ respectively. The evolution of the stochastic shocks is given by the transition function Q defined on $(Z, \mathcal{Z})$ with the Feller property. A (measurable) set $\Omega \subset X \times X \times Z$ describing the feasibility of decisions, *i.e.* if $(x, z) \in X \times Z$ are the current values of the state variable and the shock then $y \in X$ is feasible for the next period if and only if $(x, y, z) \in \Omega$. From this we can define the correspondence $\Gamma : X \times Z \to \mathbb{R}^l$ by $\Gamma(x, z) = \{y \in X;\ (x, y, z) \in \Omega\}$. The one-period return function $F : \Omega \to \mathbb{R}$ is such that $F(x, y, z)$ is the current return if y is chosen for the next period from (x, z). The discount factor is $\beta \in (0, 1)$. With all these elements, the stochastic dynamic programming problem is to find a sequence of contingent plans $(\hat{x}_t)_{t\geq 1}$ (where for all $t \geq 1$, $\hat{x}_t : Z^t \to X$ is a measurable function) such that it solves the following maximization:

$$\begin{aligned} v(x_0, z_0) = \quad & \text{Max} \sum_{t=0}^{\infty} \int_{Z^t} \beta^t F(x_t, x_{t+1}, z_t)\, Q^t(z_0, dz^t) \\ & \text{subject to } (x_t, x_{t+1}, z_t) \in \Omega \text{ for all } t \geq 0 \\ & (x_0, z_0) \in X \times Z \text{ given} \end{aligned}$$

The following hypotheses will be used in this work.

Hypothesis 1. The correspondence Γ is nonempty, compact-valued, continuous and for all $x, x' \in X$, $z \in Z$ and $\alpha \in [0,1]$ it satisfies:

$$\alpha\Gamma(x,z) + (1-\alpha)\Gamma(x',z) \subset \Gamma(\alpha x + (1-\alpha)x', z).$$

Hypothesis 2. The function F is bounded, continuous and there exists $\eta_1 > 0$ such that $F(x,y,z) + (\eta_1/2)|x|^2$ is a concave function in (x,y).

Hipothesis 1 as well as boundedness and continuity of F are quite general in models with bounded returns and technologies with non-increasing returns. The second part of hypothesis 2 (also called the strong concavity[1] hypothesis of $F(.,.,z)$) was also used by Santos and Vigo-Aguiar [9] for obtaining error bounds on their estimations. Also, Montrucchio [7] used that hypothesis for obtaining Lipschitz continuity of the policy functions. Conditions on the utility functions and technologies that ensures the strong concavity in optimal growth models are given by Venditti [12].

Under these assumptions, the value function v is well-defined and satisfies $\|v\|_\infty \leq \|F\|_\infty/(1-\beta)$. From now on, $\|\cdot\|$ stands for $\|\cdot\|_\infty$.

3 The main result

In this section we will state the main theorem that estimates the approximation error in the optimal policy function computed through the contractive method defined below. This estimate is obtained using primitive data of the problem. Let T be the operator on $C(X \times Z)$ (the set of continuous and bounded functions defined in $X \times Z$ with the topology induced by the supremum norm) defined by:

$$TV(x,z) = \operatorname*{Max}_{\{y\in X;\,(x,y,z)\in\Omega\}} F(x,y,z) + \beta \int_Z V(y,z')\, Q(z,dz').$$

It is well known (see Stokey and Lucas with Prescott [10]) that under hypotheses 1 and 2 this operator is a contraction mapping with modulus β and fixed point v (the value function). The contractive method based on this operator is defined as follows: let $v_0 \in C(X \times Z)$ be a concave function in x and define the sequence $(v_n)_{n\geq 0}$ by:

$$v_{n+1}(x,z) = Tv_n(x,z),\ \forall\,(x,z) \in X \times Z, \forall\, n \geq 0 \qquad (1)$$

[1]A function $f : C \subset \mathbb{R}^n \to \mathbb{R}$ is strongly concave if there exists $\eta > 0$ such that $f(x) + (\eta/2)|x|^2$ is a concave function. The greatest η with that property is called the parameter of strong concavity.

Lemma 3.1 *With hypotheses 1 and 2, each v_n defined by (1) is strongly concave in x with parameter η_1 for all $n \geq 1$. In particular the value function v is also strongly concave in x with the same parameter.*

Since for each $n \geq 1$, v_n is strongly concave, we can define:

$$g_n(x,z) = \underset{\{y\in X;\,(x,y,z)\in\Omega\}}{\text{Argmax}} \quad F(x,y,z) + \beta \int_Z v_n(y,z')\, Q(z,dz'), \qquad (2)$$

and the optimal policy is a function given by:

$$g(x,z) = \underset{\{y\in X;\,(x,y,z)\in\Omega\}}{\text{Argmax}} \quad F(x,y,z) + \beta \int_Z v(y,z')\, Q(z,dz').$$

The main theorem *With hypotheses 1 and 2, the sequence $(v_n, g_n)_{n\geq 1}$, defined by (1) and (2) satisfies:*

$$\|g - g_n\| \leq \left(\frac{2}{\eta_1}\|v - v_n\|\right)^{1/2}.$$

In particular,

$$\|g - g_n\| \leq \left[\frac{2}{\eta_1}\left(\frac{\|F\|}{1-\beta} + \|v_0\|\right)\right]^{1/2} \beta^{n/2}.$$

A similar result can be proved substituting Hypothesis 2 by the following:

Hypothesis 3 The function F is bounded, continuous and there exists $\eta_2 > 0$ such that $F(x,y,z) + (\eta_2/2)|y|^2$ is a concave function in (x,y).

Theorem 3.2 *With hypotheses 1 and 3, the sequence $(v_n, g_n)_{n\geq 0}$ defined by (1) and (2) satisfies:*

$$\|g - g_n\| \leq \left(\frac{2\beta}{\eta_2}\|v - v_n\|\right)^{1/2}.$$

In particular,

$$\|g - g_n\| \leq \left[\frac{2}{\eta_2}\left(\frac{\|F\|}{1-\beta} + \|v_0\|\right)\right]^{1/2} \beta^{(n+1)/2}.$$

4 Robustness of the approximation method

In this section we will show that the approximation method is robust by jointly considering errors in computing the T operator and errors in computing the maximizer in each iteration.

Suppose that T is performed using a numerical method and that $\widetilde{T}$, an "approximated" operator, is computed.

Hypothesis 4 Let $\widetilde{T} : C(X \times Z) \to C(X \times Z)$. Assume that there exists $\varepsilon \geq 0$ such that for all $f \in C(X \times Z)$

$$\|\widetilde{T}(f) - T(f)\| \leq \varepsilon.$$

Now let $(\tilde{v}_n)_{n\geq 0}$ be a sequence generated by the rule

$$\tilde{v}_{n+1} = \widetilde{T}(\tilde{v}_n).$$

Proposition 4.1 *If the correspondence* Γ *is nonempty, compact-valued,and continuous, the function* F *is bounded and continuous, and* $\widetilde{T}$ *satisfies Hypothesis 4, then the sequence* $(\tilde{v}_n)_{n\geq 0}$ *satisfies*

$$\|\tilde{v}_n - v\| \leq \frac{\varepsilon}{1-\beta} + \beta^n \left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\| \right).$$

Remark The aplication $\widetilde{T}$ does not have to satify the usual assumptions of monotonicity ($f \leq g \Rightarrow \widetilde{T}f \leq \widetilde{T}g$) and discounting ($\widetilde{T}(f+a) = \widetilde{T}f + \beta a,\ a \in \mathbb{R}$) (see Santos and Vigo [9]). Since $\widetilde{T}$ represents the "inexact" operator T these assumptions are hard to check (and may not hold) when rounding and chopping errors are embedded into the analysis.

Proposition 4.1 also says that if

$$\beta^n \left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\| \right) \ll \frac{\varepsilon}{1-\beta}$$

then more than n iterations may not appreciably improve the accuracy of the estimated value function.

In addition, suppose that an *inexact* maximization is used to compute the policy associated to $\tilde{v}_n$. That is, take a tolerance $\tau \geq 0$ and define $\widetilde{G}_n(x,z)$ as those $\tilde{y} \in \Gamma(x,z)$ such that $\forall y \in \Gamma(x,z)$

$$F(x,\tilde{y},z) + \beta \int_Z \tilde{v}_n(\tilde{y},z')Q(z,dz') \geq F(x,y,z) + \beta \int_Z \tilde{v}_n(y,z')Q(z,dz') - \tau. \tag{3}$$

Observe that $\widetilde{G}_n$ can be a correspondence. In general, practical computation does not provide *the whole* set $\widetilde{G}_n(x,z)$. Instead, the inexact maximization will provide some $\tilde{y}_n \in \widetilde{G}_n(x,z)$. Even so, we have the following estimation.

Theorem 4.2 *Suppose that hypotheses 1, 2 and 4 are satisfied. Let $\tilde{g}_n : X \times Z \to X$ be a selection of $\widetilde{G}_n$, that is, $\tilde{g}_n(x,z) \in \widetilde{G}_n(x,z)$ for all (x,z). Then,*

$$\|g - \tilde{g}_n\| \leq \left[\frac{4}{\eta_1}\|v - \tilde{v}_n\| + \tau\right]^{1/2}.$$

In particular,

$$\|g - \tilde{g}_n\| \leq \left[\frac{4}{\eta_1}\left(\frac{\varepsilon}{1-\beta} + \beta^n\left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\|\right)\right) + \tau\right]^{1/2}.$$

Remark Theorem 4.2 says that if n is such that

$$\beta^n\left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\|\right) \ll \frac{\varepsilon}{1-\beta} + \frac{\eta_1\tau}{4}$$

then more than n iterations may not appreciably improve the accuracy of the estimated policy.

5 Conclusions

In this paper we show that the contraction method defined from the Bellman equation provides accurate estimates for the optimal policy function of the stochastic dynamic programming problem. An error bound for the estimate using the n-th iteration is explicitly constructed. It only depends on the norm and the parameter of strong concavity of the return function and the discount factor. Neither differentiability of the return function nor interiority of the solution are required. This can be useful when the model involves piece-wise linear taxation/subsidies or short-run Leontief technologies because in these cases the return function may not be differentiable. Also interiority of solutions can not be guaranteed if (for example) Inada's condition is not considered.

When inexact computations are performed in the calculation of T (making a discretization of the state space, for example) or in the maximizer of each iteration we also provide an error bound for the estimation of the value and policy functions. The intuition is quite simple: Perturbations of a contraction mapping are stable even though a fixed point for the perturbated mapping may not exist.

The error bounds presented in this paper can be used for evaluating *a priori* the number of iterations needed in practical computations of the Bellman method. For example, following the notation of Section 4, let ε be the error on the T operator and τ be the error on the maximization procedure used to compute the associated policy. If n is such that

$$\beta^n \left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\| \right) \ll \frac{\varepsilon}{1-\beta} + \frac{\eta_1 \tau}{4}$$

then more than n iterations may not appreciably improve the accuracy of the estimated policy.

Finally, it is important to note that the error bounds found in this work can be used combining the Bellman method with other methods (which in general are faster than the Bellman one). Suppose that it is found an approximation for the value function $\hat{v}_0$ by any method and let $\hat{v}_1 = T\hat{v}_0$. Using the inequality $\|v - \hat{v}_0\| \leq \|v - \hat{v}_1\| + \|\hat{v}_1 - \hat{v}_0\|$, the contraction property of T and the main theorem we will obtain:

$$\|g - \hat{g}_0\| \leq \left(\frac{2}{\eta_1(1-\beta)} \|\hat{v}_1 - \hat{v}_0\| \right)^{1/2},$$

where

$$\hat{g}_0(x,z) = \underset{\{y \in X;\, (x,y,z) \in \Omega\}}{\text{Argmax}} \quad F(x,y,z) + \beta \int_Z \hat{v}_0(y,z') \, Q(z,dz').$$

A Appendix

To prove Lemma 3.1, let us show the following:

Lemma A.1 *$F(x,y,z) + (\eta/2)|x|^2$ is a concave function in (x,y) if and only if for all $(x_i, y_i, z) \in \Omega$, $i = 1,2$ and for all $\alpha \in [0,1]$ it holds that:*

$$F(x^\alpha, y^\alpha, z) \geq \alpha F(x_1, y_1, z) + (1-\alpha) F(x_2, y_2, z) + \frac{\eta}{2}\alpha(1-\alpha)|x_1 - x_2|^2,$$

where $x^\alpha = \alpha x_1 + (1-\alpha)x_2$ and $y^\alpha = \alpha y_1 + (1-\alpha)y_2$.

Proof:
($\Rightarrow$) By hypothesis:

$$F(x^\alpha, y^\alpha, z) + \frac{\eta}{2}|x^\alpha|^2 \geq \alpha[F(x_1,y_1,z) + \frac{\eta}{2}|x_1|^2] + (1-\alpha)[F(x_2,y_2,z) + \frac{\eta}{2}|x_2|^2],$$

expanding the square of the left side and simplifying it results in:

$$F(x^\alpha, y^\alpha, z) \geq \alpha F(x_1, y_1, z) + (1-\alpha)F(x_2, y_2, z) + \frac{\eta}{2}\alpha(1-\alpha)|x_1 - x_2|^2.$$

($\Leftarrow$) Completely analogous.

Proof of Lemma 3.1 It will be sufficient to prove that if $V(., z)$ is a concave function then $TV(., z)$ is a strongly concave function with parameter η_1. Let $x_1, x_2 \in X$, $\alpha \in [0,1]$, $x^\alpha = \alpha x_1 + (1-\alpha)x_2$ and for $i = 1, 2$ let $y_i \in \Gamma(x_i, z)$ be such that:

$$TV(x_i, z) = F(x_i, y_i, z) + \beta \int_Z V(y_i, z')\, Q(z, dz').$$

Then, using hypotheses 1, 2 and Lemma A.1 we have that:

$$\begin{aligned} TV(x^\alpha) &\geq F(x^\alpha, \alpha y_1 + (1-\alpha)y_2, z) + \beta \int_Z V(\alpha y_1 + (1-\alpha)y_2, z')Q(z, dz') \\ &\geq \alpha F(x_1, y_1, z) + (1-\alpha)F(x_2, y_2, z) + \frac{\eta}{2}\alpha(1-\alpha)|x_1 - x_2|^2 + \\ &\qquad \beta \int_Z [\alpha V(y_1, z') + (1-\alpha)V(y_2, z')]\, Q(z, dz') \\ &= \alpha TV(x_1, z) + (1-\alpha)TV(x_2, z) + \frac{\eta}{2}\alpha(1-\alpha)|x_1 - x_2|^2. \end{aligned}$$

Since the set of strongly concave functions is a closed set with the topology induced by the sup norm it follows that the fixed point of T is strongly concave.

To prove the main theorem, we will need the following lemmata.

Lemma A.2 *$f : C \subset \mathbb{R}^n \to \mathbb{R}$ (C is a convex set) is strongly concave with modulus η if and only if for all $x_1, x_2 \in C$ it holds that:*

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)f(x_2) + (\eta/2)\alpha(1-\alpha)|x_1 - x_2|^2.$$

Proof: Analogous to the proof of lemma A.1.

Lemma A.3 *Let $f : C \subset \mathbb{R}^n \to \mathbb{R}$ (C is a convex set) be a strongly concave function with modulus η. If $x^* = Argmax_{x \in C}\, f(x)$ then*

$$f(x) \leq f(x^*) - \frac{\eta}{2}|x - x^*|^2, \quad \forall x \in C.$$

Proof: Let $x \in C$ and $\alpha \in (0,1)$. By definition of x^* and the characterization of strong concavity given in Lemma A.2 we have:

$$f(x^*) \geq f(\alpha x^* + (1-\alpha)x) \geq \alpha f(x^*) + (1-\alpha) f(x) + \tfrac{\eta}{2}\alpha(1-\alpha)|x-x^*|^2$$

$$\Rightarrow f(x^*) \geq f(x) + \tfrac{\eta}{2}\alpha|x-x^*|^2,$$

making $\alpha \to 1$ we obtain the result.

Proof of the main theorem.

Let us fix some notations. Let $v_n \in C(X \times Z)$ be the n-th iteration ($n \geq 1$) of the T operator from some initial concave function $v_0 \in C(X \times Z)$. Let

$$g_n(x,z) = \text{Argmax}_{\{y \in X;\, (x,y,z) \in \Omega\}} F(x,y,z) + \beta \int_Z v_n(y,z') \, Q(z,dz'),$$

$$\phi_n(x,y,z) = F(x,y,z) + \beta \int_Z v_n(y,z',) \, Q(z,dz'),$$

$$\phi(x,y,z) = F(x,y,z) + \beta \int_Z v(y,z') \, Q(z,dz').$$

By lemma 3.1 the functions $\phi(x,.,z)$ and $\phi_n(x,.,z)$ are strongly concave with parameter $\beta\eta_1$. Then by lemma A.3 we have that:

$$\phi(x,g(x,z),z) \geq \phi(x,g_n(x,z),z) + \frac{\beta\eta_1}{2}|g(x,z) - g_n(x,z)|^2,$$

$$\phi_n(x,g_n(x,z),z) \geq \phi_n(x,g(x,z),z) + \frac{\beta\eta_1}{2}|g(x,z) - g_n(x,z)|^2.$$

Summing up the above inequalities we obtain that:

$$\beta\{\int_Z [(v_n - v)(g_n(x,z),z') + (v - v_n)(g(x,z),z')] \, Q(z,dz')\} \geq \beta\eta_1 |g(x,z) - g_n(x,z)|^2$$

$$\Rightarrow 2\|v - v_n\| \geq \eta_1 |g(x,z) - g_n(x,z)|^2$$

this inequality holds for all $(x,z) \in X \times Z$, so we conclude:

$$\|g - g_n\| \leq [\frac{2}{\eta_1}\|v - v_n\|]^{1/2}.$$

Finally, using that T is a contraction it results that $\|v - v_n\| \leq \beta^n \|v - v_0\|$. The bound is obtained since $\|v\| \leq \|F\|/(1-\beta)$.

Proof of theorem 3.2

Under hypothesis 3, $\phi_n(x,.,z)$ and $\phi(x,.,z)$ are strongly concave with parameter η_2. Then by lemma A.3 we have that:

$$\phi(x,g(x,z),z) \geq \phi(x,g_n(x,z),z) + \frac{\eta_2}{2}|g(x,z) - g_n(x,z)|^2,$$

$$\phi_n(x,g_n(x,z),z) \geq \phi_n(x,g(x,z),z) + \frac{\eta_2}{2}|g(x,z) - g_n(x,z)|^2.$$

Using the same reasoning as in the proof of the main theorem we conclude that:

$$\|g - g_n\| \leq [\frac{2\beta}{\eta_2}\|v - v_n\|]^{1/2} \leq \left[\frac{2}{\eta_2}\left(\frac{\|F\|}{1-\beta} + \|v_0\|\right)\right]^{1/2} \beta^{(n+1)/2}.$$

Proof of Proposition 4.1

By our assumptions, T is a β-contraction on $C(X \times Z)$ with fixed point v. Using also the definition of the sequence $(\tilde{v}_n)_{n\geq 0}$, the triangular inequality and Hypothesis 4 we get

$$\begin{aligned} \|\tilde{v}_{n+1} - v\| &= \|\widetilde{T}(\tilde{v}_n) - v\| \\ &\leq \|\widetilde{T}(\tilde{v}_n) - T(\tilde{v}_n)\| + \|T(\tilde{v}_n) - v\| \\ &\leq \varepsilon + \beta\|\tilde{v}_n - v\|. \end{aligned}$$

Hence

$$\|\tilde{v}_n - v\| \leq \sum_{j=0}^{n-1} \varepsilon\beta^j + \beta^n\|\tilde{v}_0 - v\| \leq \varepsilon/(1-\beta) + \beta^n\|\tilde{v}_0 - v\|.$$

Proof of theorem 4.2

Let

$$\tilde{\phi}_n(x,y,z) = F(x,y,z) + \beta\int_Z \tilde{v}_n(y,z')\, Q(z,dz').$$

Then

$$\tilde{\phi}_n(x,\tilde{g}_n(x,z),z) \geq \tilde{\phi}_n(x,g(x,z),z) - \tau.$$

With hypotheses 1 and 2 we have:

$$\phi(x,g(x,z),z) \geq \phi(x,\tilde{g}_n(x,z),z) + \frac{\beta\eta_1}{2}|g(x,z) - \tilde{g}_n(x,z)|^2.$$

Adding up these inequalities and following the same procedure as in the proof of the main theorem we will obtain

$$\|g - \tilde{g}_n\| \leq \left[\frac{4}{\eta_1} \|v - \tilde{v}_n\| + \tau \right]^{1/2}.$$

Finally, using Proposition 4.1 it will result that

$$\|g - \tilde{g}_n\| \leq \left[\frac{4}{\eta_1} \left(\frac{\varepsilon}{1-\beta} + \beta^n \left(\frac{\|F\|}{1-\beta} + \|\tilde{v}_0\| \right) \right) + \tau \right]^{1/2}.$$

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