Equilibrium Option Pricing and Market Incompleteness Driven by Illiquidity

João Amaro de Matos*and Paula Antão
Faculdade de Economia
Universidade Nova de Lisboa
Rua Marquês de Fronteira 20
1099-038 Lisbon, Portugal

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Abstract

Under perfect market equilibrium, option prices are determined as if the economic agents were risk neutral. This paper develops a simple two-period model to analyze the impact of imperfect hedging on the equilibrium pricing of derivatives in the presence of a monopolistic market-maker. In a partial equilibrium analysis, we shown how a bid-ask option price spread is generated. In particular, we show how the equilibrium bid and ask prices depend on the market-maker's risk aversion and also on the demand and supply curves for options. Neither inventory costs nor asymmetric information are considered.

1 Introduction

Among the traditional assumptions on which option pricing is based, markets are perfect and the underlying asset can be transacted at any point in time. Under the absence of arbitrage opportunities the value of an option can be computed as the value of a portfolio on the underlying risky asset and risk-free bonds that exactly replicates the payoffs of the derivative. This portfolio can be rebalanced in a self-financing way

*Corresponding author's contact: amatos@fe.unl.pt.

until the maturity of the option, by continuously transacting the underlying asset and the bonds. Under these assumptions, the calculated value of the initial portfolio can be shown to be the equilibrium price of the option. In this paper we assume that the underlying asset cannot be transacted at every point in time and study the impact of this constraint on the equilibrium pricing of options.

The fact that the underlying asset can be transacted only at some points in time can be described as a lack of liquidity of the market for the underlying asset as in Longstaff (2001). This illiquidity implies that the markets become incomplete in the sense that perfect hedging of the option in all states of nature is no longer possible. However, for any given option, portfolios can be found that have the same payoff as the option in some states of nature and higher payoffs in the other states. Such portfolios are said to be *superreplicating*. Holding one such portfolio should be worth more than the option itself and therefore, the value of the cheapest of such portfolios should be seen as a bound on the value of the option. The nature of the superreplicating bounds is well characterized in the context of incomplete markets in the papers by El Karoui and Quenez (1991) and (1995), Edirisinghe, Naik and Uppal (1993) and Karatzas and Kou (1996). As stressed by these authors, under market incompleteness the hedging of the market-maker is different depending on whether he is in a long or a short position. This results in a lower and an upper bound for the option prices.

We introduce a two-period binomial model where the underlying asset cannot be transacted in the intermediate point in time. The superreplicating bounds for a short and long position in an option are derived. These values, however are not necessarily the market price of the options. They are the bounds for the price consistent with non-negative profits for the seller or the buyer of an option. We introduce in our model a risk-averse monopolist market-maker willing to accept some negative profits when setting bid and ask prices. We characterize the equilibrium behavior of the intermediary under this type of market incompleteness and show how it generates a bid-ask spread for the market prices at which options are transacted.

Our work is related to the failure of the continuous hedging portfolio rebalancing hypothesis. This hypothesis has been dropped in several previous studies, mainly because of the existence of transaction costs[1]. In the presence of transaction costs, the assumption of continuous rebalancing is no longer reasonable because the replicating strategy would be extremely costly. In order to take the impact of this imperfection on the price into account, most authors assume that trading takes place only between discrete time intervals. This approach also generates market incompleteness, leading naturally to bounds for equilibrium option prices, although of a different nature of those described here.

The superreplicating bounds establish the limits of the interval for the prices outside which the market-maker has a positive profit with probability one. In other words, an arbitrage opportunity exists if the market-maker sells options above the upper bound or buys options below the lower bound. Clearly, this approach does not determine equilibrium prices. Considering the possibility of transacting within the superreplicating bounds, the market-maker can no longer replicate exactly the payoff of the option and there are no arbitrage opportunities. In particular, there is

[1] Leland (1985), Merton (1990), Boyle and Vorst (1992) and Toft (1996), among others, have examined the impact of transaction costs on option valuation by studying the construction of hedging strategies that replicate the outcomes from options. Hodges and Neuberger (1989), Davis, Panas and Zariphopoulou (1993) and Constantinides and Zariphopoulou (1999) also studied the problem of transaction costs via a utility maximization approach.

some risk involved. For this reason, the attitude towards risk of buyers and sellers must be considered. Hodges and Neuberger (1989), Davis, Panas and Zariphopoulou (1993), Rouge and El Karoui (2000) and Frittelli (2000), among others, also studied some aspects of the equilibrium pricing of options in the framework of incomplete markets. Their models assume the existence of small investors with the opportunity to transact a derivative asset. The option prices are obtained by comparing maximized utilities with and without the opportunity to transact the option. However, none of these papers studies the interaction between incompleteness and the structure of the market. Here, the presence of a monopolistic market-maker and the risk-aversion of the agents involved play a central role.

The explanation for the existence of a bid-ask spread[2] for options in this work is quite different from the explanations in the traditional market microstructure literature. Microstructure models can be broadly classified as inventory models and asymmetric information models. Inventory models[3] consider risk-averse market-makers who set a bid-ask spread to compensate intermediaries for bearing undesired inventory. When there is an order imbalance that moves the market-maker away from his desired inventory position, he adjusts the bid-ask spread to attract orders to move back to his optimal inventory position. Information asymmetry models[4] assume that

[2] Although there is much work on stock bid-ask spreads, the spread of options has been investigated by fewer researchers. Biais, Foucault and Salanié (1998) analyze three different market structures and the ways that the associated restrictions lead to differences in prices, bid-ask spreads, trades and risk-sharing. There are also a few empirical studies that examine bid-ask spreads in the option markets, such as, George and Longstaff (1993), Chan, Chung and Johnson (1995) and Etling and Miller, jr (2000).

[3] Among others, Stoll (1978) and Amihud and Mendelson (1980) studied bid-ask spreads and stock inventory. More recently, Lee, Mucklow and Ready (1993), Hasbrouck and Sofianos (1993), Madhavan and Smidt (1993) and Manaster and Mann (1996) also found some evidence on the relationship of bid-ask spreads to dealer inventory control costs.

[4] Some authors discussing this topic: Copeland and Galai (1983), Glosten and Milgrom (1985), Admati and Pfleiderer (1992) and Foster and Viswanathan (1994).

an adverse selection problem exists because the market-maker is at an informational disadvantage to the informed traders. In this case spreads must be kept wide enough to ensure that gains from trading with the uninformed agents exceed the losses associated with trading with informed agents.

Our model assumes neither asymmetric information nor optimal inventory strategy and still explains the existence of an equilibrium bid-ask spread. It assumes an intermediary with monopolistic market power and also assumes that all transactions of options should be made through the market-maker. The monopolistic nature of the market-maker can be justified in some markets or, at least, in some trading periods. For example, in markets such as the NYSE there are monopolist specialists. On the other hand, authors such as Brock and Kleidon (1992) show that demand at the opening and closing of markets is greater and less elastic than at other times of the trading day. In markets where a single specialist market-maker has monopolistic power, these authors provide evidence that during the opening and closing of the markets the specialist uses his market power in order to charge higher spreads.

Another interesting point is that we consider a market-maker with no optimal inventory policy. In general, intermediaries hold inventories of goods on hand and stand ready to sell to costumers. They also have cash on hand and stand ready to buy from suppliers. This avoids the problem of coincidence of wants. However, when talking about options there is not the problem of holding inventories. If the quantity of options sold is different from the quantity bought, the market-maker constructs some hedging portfolio to cover this difference and there is no need to hold a physical inventory of options.

Our results are as follows. (1) Equilibrium prices are within the superreplicating bounds. (2) The bid-ask spread is shown to depend on (a) the degree of the market-maker's risk aversion and (b) the elasticities of both the demand and supply curves for options. (3) Finally, in the specific case of a negative exponential utility function we characterize the necessary conditions for prices to coincide with the bounds. These conditions are more likely to be satisfied as the rigidity of the demand and supply market curves increases. On the other hand, when the market-maker is infinitely risk-averse, the market-maker prefers not to hold any hedging portfolio, selling and buying exactly the same number of option contracts.

This article is organized as follows. Section 2 introduces incompleteness in the two-period model driven by illiquidity, i.e., by the fact that the underlying asset cannot be transacted at some points in time. In Section 3 the superreplicating portfolios are derived and the corresponding options pricing bounds are obtained. Section 4 considers a risk-averse market-maker who is willing to accept some negative profits in the future, and we also study how the market-maker decides between complete or partial hedging. As expected, the cost of the hedging portfolios that do not cover all states of nature are less expensive, allowing for a lower ask price and higher bid price. Finally, Sections 5 and **??** consider the demand and supply curves and the behavior of a monopoly market-maker, characterizing the equilibrium bid and ask prices. The last section presents the main conclusions. All proofs that are not in the text are presented in the appendix.

2 An Incomplete Two-Period Model

This work is based on the well-known binomial option pricing model developed by Cox, Ross and Rubinstein (1979). In this model the stock price follows a binomial process over discrete periods. Let the initial value of the stock be denoted by S. After each period the value of the stock can assume two possible values as compared to the beginning of the period: it can be multiplied either by a rate U or by a rate D, where $U > R > D$ and R denotes the riskless total return over each time period.

In this simplified model there are three relevant dates: $t = 0, 1, 2$. At $t = 2$ there are three possible states of nature, depending on how many times the original value of the underlying asset has been multiplied by D after two time periods. Then, the three possible states of nature are labeled by $i = 1, 2, 3$ and for each i, the state is characterized by the value

$$S_{2,i} = SU^{3-i}D^{i-1}$$

for the stock at time $t = 2$.

Consider a European call option with exercise price K and two periods to maturity. At $t = 0$ the option is traded for a value C, and at $t = 2$ the option matures. Its value at maturity is given by

$$C_{2,i} = \max(0, S_{2,i} - K),$$

for the three possible states of nature $i = 1, 2, 3$.

2.1 The Complete Two-Period Model

If there are no arbitrage opportunities, a call option must be worth the same as the cheapest portfolio that exactly replicates the value of the call at each point in time. Considering a simplified economy with one risky asset (the underlying) and one-period riskless bonds, such a portfolio may be constructed. At each point in time this portfolio consists of Δ shares of the stock and an amount B in riskless bonds. As time changes, the portfolio is adjusted to continue replicating the values of the call option. Then, at $t = 0$, a portfolio of Δ_0 shares and an amount B_0 in the riskless asset is built such that it replicates the value of the call at $t = 1$. At $t = 1$ the portfolio is adjusted to replicate the option at maturity. For longer maturities, this implies that every time the price changes, there may be transactions in the market of the underlying stock and in the market of bonds to adjust the hedging portfolio.

If there are no arbitrage opportunities, the value of the call option at each point in time and at each possible state of nature must be the same as the value of this corresponding hedging portfolio, since both the call and the portfolio have exactly the same payoffs at the next point in time.

For this two-period model, it is well known that the value of the call is

$$C = \left[\pi^2 C_{2,1} + 2\pi(1-\pi)C_{2,2} + (1-\pi)^2 C_{2,3}\right]/R^2$$

where $\pi = \frac{R-D}{U-D}$ and $1 - \pi = \frac{U-R}{U-D}$.

2.2 Modeling Incompleteness

At this point we introduce the notion of incompleteness. For this purpose let us assume that at $t = 1$ trading in the underlying asset is not possible. In other words, it is not possible to convert money into the asset, or vice-versa, in every period. In this specific case, it will take two periods of time to achieve these conversions. This trading restriction will give rise to an incomplete market, as shown below. It is also consistent with the characteristics of actual financial markets where it may take an extended period of time to transact an asset.

Liquidity has traditionally been measured in terms of bid-ask spreads or transaction costs. However, in this work, as in Longstaff (2000), liquidity is more related to the quantity of trades that can be executed than with the costs of trading. Hence, illiquidity exists because the ability to buy or sell securities, at any price, is restricted. In our model, this restriction occurs at time $t = 1$ since it is imposed that it is impossible to transact at this date. The point of this paper is to understand how this fact affects the pricing of the option.

To introduce the above trading restriction in the pricing model, it is assumed that at $t = 1$ there is no trading in the underlying stock. This implies that the initial portfolio (Δ_0, B_0) made at $t = 0$ will not be adjusted after one period. An investor who buys this portfolio (Δ_0, B_0) and keeps it until maturity is not fully hedged because this portfolio does not consider the evolution of the stock price until time $t = 2$. On the other hand, another portfolio could have been computed considering the three states of nature at $t = 2$. Once again, a portfolio computed as in Section 2.1 would not replicate all the payoffs of the call option at maturity since it could have the same payoffs as the call option only in two of the three alternative states of nature. It follows that the usual approach to compute replicating portfolios may lead to a future situation where the market-maker may have to support some negative profits. The evolution of the underlying asset now follows a trinomial tree as can be seen in Figure 1.

3 The fully hedged position

3.1 The minimum selling price of a call option

In this section we consider that financial institutions trading options construct superreplicating portfolios, that is, portfolios that give the institution a null or positive payoff at maturity. The cost of the superreplicating portfolios corresponds to the

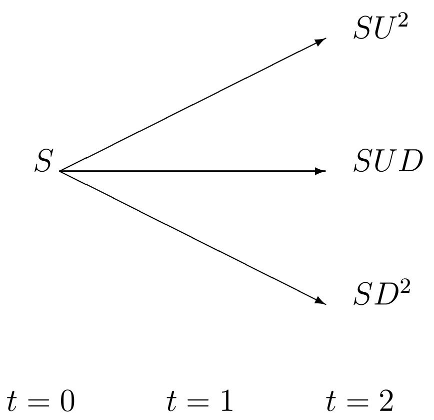

Figure 1: The trinomial tree for the underlying asset prices

maximum and minimum prices for an option consistent with a non-negative profit.

Consider a financial institution selling a call option while wishing to be hedged. Today, the objective of the institution is to minimize the cost of replicating the exercise value of the option at maturity. It is expected that the buyer of the call option exercises it, at maturity, whenever the value of the stock is greater than the exercise price. The replicating portfolio is built in such a way that at $t = 2$ its value always exceeds, or equals, the exercise value of the option. In other words, the financial institution must be prepared for the exercise of the option. If this happens, the institution will certainly need to hold an instrument that is worth at least as much as the exercise value of the option. Therefore, the profit function of the financial institution is given by

$$\begin{aligned}\Pi_0^a &= C - (\Delta S + B), \text{ at } t = 0 \\ \Pi_{2,i}^a &= (\Delta S_{2,i} + BR^2) - C_{2,i} \text{ at } t = 2, \text{ for } i = 1, 2, 3.\end{aligned}$$

The problem of the intermediary is to minimize the cost of this initial portfolio, that is, the following optimization problem must be solved

$$\min_{\{\Delta, B\}} \Delta S + B$$

subject to the terminal conditions:

$$\Delta S_{2,i} + BR^2 \geq C_{2,i} \text{ for } i = 1, 2, 3.$$

where $S_{2,i}$ and $C_{2,i}$ have the meanings explained at the beginning of Section 2.

Notice that these terminal restrictions are inequalities. It follows that the solution of this problem will clearly be superreplicating, since there cannot be found a unique solution for (Δ, B) satisfying the three conditions as equalities. In at least one of the three states of nature, the portfolio will have a strictly higher payoff than the exercise value of the option. With a portfolio guaranteeing these restrictions, the institution is simultaneously maximizing its profits while being fully hedged against the exercise of the option.

Proposition 1 *The solution to this problem is obtained with a portfolio of $\overline{\Delta}$ shares and an amount $\overline{B}$ invested in the risk-free asset given by*

$$\overline{\Delta} = \frac{C_{2,1} - C_{2,3}}{S(U^2 - D^2)},$$
$$\overline{B} = \frac{U^2 C_{2,3} - D^2 C_{2,1}}{R^2(U^2 - D^2)},$$

leading to a cost $\overline{C} = \frac{1}{R^2}\left[\frac{R^2 - D^2}{U^2 - D^2} C_{2,1} + \frac{U^2 - R^2}{U^2 - D^2} C_{2,3}\right]$.

Proof. *See the Appendix.* ■

It is easy to check that the cost of the superreplicating portfolio $(\overline{\Delta}, \overline{B})$ is always greater than or equal to the cost of a hedging portfolio when the stock market is liquid. In other words, the financial institution has a lower bound on its selling price that is higher than in the case of full liquidity. Finally, the profit of the institution at maturity is given by

$$\overline{\Pi}_{2,1} = 0$$

$$\begin{aligned}\overline{\Pi}_{2,2} &= \frac{U}{D+U}\left(K - D^2 S\right) > 0, \text{ if } UDS > K \geq SD^2 \\ &= \frac{D}{U+D}(U^2 S - K) > 0, \text{ if } U^2 S > K \geq SDU \\ &= 0, \text{ otherwise.}\end{aligned}$$

$$\overline{\Pi}_{2,3} = 0$$

In order to construct such a hedging portfolio, the intermediary must sell the call option at a price $C^a \geq \overline{C}$. Otherwise, the institution does not have enough resources to build the portfolio. Notice that having once chosen this portfolio, the profit at time $t = 2$ is positive and constant, that is, it is not a function of the call option price at $t = 0$. The profit at time $t = 0$ is null when $C^a = \overline{C}$ and strictly positive if $C^a > \overline{C}$.

3.2 The maximum buying price of a call option

The financial institution is also concerned about the cost of replicating a long call option on the same underlying asset. The problem now is analogous but quite different. The financial institution buys a call option and, to be hedged, sells a hedging portfolio. This is different from the previous case since it corresponds to short selling the underlying asset while investing in the riskless asset. At maturity, the institution must buy the shares in the market and receives the results from the investment in the riskless asset. At the same time, it holds a call option that is exercised if the payoff is positive.

The profit function of the institution is now given by

$$\Pi_0^b = (\Delta S + B) - C, \text{ at } t = 0$$
$$\Pi_{2,i}^b = C_{2,i} - (\Delta S_{2,i} + BR^2) \text{ at } t = 2, \text{ and } i = 1, 2, 3.$$

To be fully hedged against the probability of exercising the option, it must be imposed that the call option's payoff be equal to at least as much as the value of the portfolio. At the same time, and to maximize its profits, the financial institution searches today for the maximum value of the portfolio that it is selling. This corresponds to the following problem

$$\max_{\{\Delta, B\}} \Delta S + B$$

subject to the terminal conditions:

$$C_{2,i} \geq \Delta S_{2,i} + BR^2 for i = 1, 2, 3.$$

where $S_{2,i}$ and $C_{2,i}$ have the meanings explained at the beginning of section 2.

Once again, these terminal conditions are inequalities. The replicating portfolio will satisfy the restrictions as equalities only for two of the three states of nature. This problem has two different solutions, depending on whether $(R^2 - UD)$ is positive or negative[5]. We then have the following.

Proposition 2 *The solution of this problem, when $R^2 > UD$, is obtained for a portfolio $(\underline{\Delta}, \underline{B})$ given by*

$$\underline{\Delta} = \frac{C_{2,1} - C_{2,2}}{S(U^2 - UD)},$$
$$\underline{B} = \frac{U^2 C_{2,2} - UDC_{2,1}}{R^2(U^2 - UD)},$$

[5]For simplicity, the case $R^2 < UD$ is presented in the appendix since the results are similar.

and costs $\underline{C} = \frac{1}{R^2}\left[\frac{R^2-UD}{U^2-UD}C_{2,1} + \frac{U^2-R^2}{U^2-UD}C_{2,2}\right]$.

Proof. *See the Appendix.* ■

Choosing this portfolio, the financial institution is fully hedged in the sense that, at maturity, the payoff of the option will be greater than or equal to the cost of buying the hedging portfolio. Denoting by $\underline{\Pi}$ the profit resulting from this hedging portfolio, it is easy to verify that at time $t = 2$

$$
\begin{aligned}
\underline{\Pi}_{2,1} &= 0 \\
\underline{\Pi}_{2,2} &= 0 \\
\underline{\Pi}_{2,3} &= K - D^2 S > 0, \text{ if } UDS > K \geq SD^2 \\
&= \frac{D}{U+D}(U^2 S - K) > 0, \text{ if } U^2 S > K \geq SDU \\
&= 0, \text{ otherwise.}
\end{aligned}
$$

>From these results, one can say that if financial institutions wish to be fully hedged, they sell options at a price $C^a \geq \overline{C}$ and buy options at a price $C^b \leq \underline{C}$. The difference $C^a - C^b$ is called the bid-ask spread, and $\overline{C}$ and $\underline{C}$ are, respectively, called the ask and bid bounds.

The next section analyzes the situation of incomplete hedging, that is, situations when profits at time $t = 2$ may be negative in some states of nature.

4 Incomplete Hedging

The previous section followed the superreplication approach, that is, a positive profit in all three states of nature at maturity was imposed. Suppose now that the financial institution is willing to accept negative profits in one of the three states of nature at time $t = 2$. The hedging portfolio is expected to be less expensive, allowing to sell (buy) the call option for a lower (higher) price than $\overline{C}$ ($\underline{C}$) while keeping a null or positive profit at time $t = 0$. In the simple two-period model, the negative payoff will occur only in one state of nature at time $t = 2$. To study these situations we assume that financial institutions do not take into account *the profit* from trading options but *the utility* derived from the profit at each date. In this way, we may now incorporate the risk-averse behavior of financial institutions.

The rest of this work assumes that a financial institution has a utility function characterized as

$$\mathcal{U}(\Pi_0, \Pi_2) = \Pi_0 + v(\Pi_2)$$

where v is continuous, differentiable, strictly increasing and strictly concave. Preferences are represented by time-additive and state independent von Neumann-Morgenstern utility functions. In Section 3, the hedging portfolio was computed imposing a positive value for Π_2, in all states of nature. In this section, we consider the construction of a hedging portfolio covering only two of the three states of nature.

4.1 The selling price

Let the financial institution construct a portfolio covering only the states of nature $i = 1$ and $i = 2$. These two states of nature represent the highest losses that the institution can have since $C_{2,3} \geq C_{2,2} \geq C_{2,1}$. Covering only the states of nature corresponding to $i = 1$ and $i = 2$ is equivalent to imposing that profits in these states will be zero, allowing $\Pi_{2,3}$ to be negative. The problem to solve is to choose Δ and B such that

$$\Delta S_{2,1} + BR^2 = C_{2,1}$$

$$\Delta S_{2,2} + BR^2 = C_{2,2}.$$

This problem is solved for a portfolio with $\Delta = \underline{\Delta}$ and $B = \underline{B}$ defined above in Proposition 2. The cost of the portfolio is the value $\underline{C}$ given in the same Proposition and, as expected, it can be shown[6] that $\underline{C} < \overline{C}$. The use of this hedging portfolio leads to a negative profit if the state of nature corresponding to $i = 3$ occurs at maturity. However, this partial hedging strategy allows one to sell the option at a price C^a lower than $\overline{C}$. Also, and since the cost of building this portfolio $(\underline{\Delta}, \underline{B})$ is $\underline{C}$, the transaction value of the call must satisfy $C^a \geq \underline{C}$, so that there are resources to build the hedging strategy. Notice that the profits at time $t = 2$ depend only on the hedging portfolio and are simply given by $\Pi^a = -\underline{\Pi}$.

As a hedging alternative, the institution could decide to construct a different portfolio covering only the states of nature $i = 1$ and $i = 3$. The relevant portfolio can be easily seen to be the same $(\overline{\Delta}, \overline{B})$ described in Proposition 1. In that case, the selling price C^a must be higher than the portfolio's cost $\overline{C}$. This last alternative is the same thing as having a complete hedging.

The decision between hedging portfolios can thus be mapped into a decision between regions for the selling price C^a. Either the institution hedges with portfolio $(\underline{\Delta}, \underline{B})$ and the selling price then satisfies $\overline{C} > C^a \geq \underline{C}$, or the institution prefers to be fully hedged with portfolio $(\overline{\Delta}, \overline{B})$, leading to $C^a \geq \overline{C}$. In order to fully characterize this decision, the expected utility driven from each of the alternatives must be compared.

[6] In fact, we can obtain $\underline{C} = \overline{C}$, although this occurs only in less interesting cases. It occurs when the option is either deep in the money ($D^2S > K$) or deep out of the money ($K > U^2S$). In the former case Δ equals one, meaning that the impossibility of transacting in the underlying asset is not relevant. In the latter, the value of the option is trivially zero. In that case, there is only one price for the option which is $\underline{C} = \overline{C}$.

The expected utility from a complete hedging is given by

$$
\begin{aligned}
E\mathcal{U}^{com}(\Pi_0,\Pi_2) &= \Pi_0 + Ev(\Pi_2) \\
&= C^a - \overline{C} + p^2 v(0) + 2p(1-p)v(\overline{\Pi}_{2,2}) + (1-p)^2 v(0), \qquad (1)
\end{aligned}
$$

with $C^a \geq \overline{C}$ and where p is the true probability of the stock price being multiplied by the rate U at time $t = 1$. On the other hand, the expected utility from the portfolio $(\underline{\Delta}, \underline{B})$ providing a partial hedging is given by

$$E\mathcal{U}^{inc}(\Pi_0,\Pi_2) = C^a - \underline{C} + p^2 v(0) + 2p(1-p)v(0) + (1-p)^2 v(-\underline{\Pi}_{2,3}),$$

in the domain $\underline{C} \leq C^a < \overline{C}$.

A sufficient condition for choosing full hedging is that the worst possible value of $E\mathcal{U}^{comp}(\Pi_0,\Pi_2)$ is still better than the best possible value of $E\mathcal{U}^{inc}(\Pi_0,\Pi_2)$ or

$$E\mathcal{U}^{com}(\Pi_0,\Pi_2,C^a = \overline{C}) \geq E\mathcal{U}^{inc}(\Pi_0,\Pi_2,C^a = \overline{C}). \qquad (2)$$

Since $E\mathcal{U}(\Pi_0,\Pi_2)$ is monotonic in C^a, this is also a necessary condition leading to

Proposition 3 *The financial institution chooses to sell the option at a price* $C \geq \overline{C}$ *if and only if*

$$\overline{C} - \underline{C} \leq (1-p)\left[(1-3p)\,v(0) - (1-p)v(-\underline{\Pi}_{2,3}) + 2pv(\overline{\Pi}_{2,2})\right]. \qquad (3)$$

Proof. The result follows directly from inequality (2) and the fact that profits are monotonic in C^a. ■

Consider a risk-neutral institution, such that $v(x) = x$, thinking of a partial hedging. Noticing that the time $t = 0$ profit $C^a - \underline{C}$ is less than $\overline{C} - \underline{C}$, the above sufficient condition to give up partial hedging and set prices higher than $\overline{C}$ simplifies to

$$\overline{C} - \underline{C} \leq \frac{(1-p)}{R^2}\left(K - D^2 S\right)\left(1 + p\frac{U-D}{D+U}\right), \text{if } UDS > K \geq SD^2$$
$$\overline{C} - \underline{C} \leq \frac{(1-p)}{R^2}\left(U^2 S - K\right) D\frac{1+p}{D+U}, \text{if } U^2 S > K \geq SDU.$$

This means that if the profit at time $t = 0$ is low, it is better to set the price at or above the upper bound. In other words, a necessary condition for the institution to choose prices in the interval $[\underline{C}, \overline{C})$ is that profits at time $t = 0$ are large enough to compensate for the expected losses at time $t = 2$.

4.2 The buying price

The analysis of the buying position is not as straightforward as the previous one. In Section 3 the bid bound price was obtained when $R^2 > UD$. In that case, profit at time $t = 2$ is null in states $i = 1$ and $i = 2$ and positive in state $i = 3$. However, when $R^2 < UD$ the bound would be a different one, denoted by $\underline{\underline{C}}$, and characterized in the appendix. In this last case, profit would be positive only in the state of nature $i = 1$.

Incomplete hedging corresponds to accepting a negative profit in one of the three states of nature. In this way, it is possible to compute the price that the institution is willing to pay to buy an option which is certainly higher than each of the bound prices. As above, we focus here only on the case $R^2 > UD$ and leave the other situation for the appendix. It can be shown that when $R^2 > UD$ one obtains $\underline{C} > \underline{\underline{C}}$. Then, $\underline{\underline{C}}$ does not correspond to the portfolio that we are looking for in the sense that its cost is lower than $\underline{C}$. In fact, it is now relevant to consider only the construction of a portfolio covering the states of nature $i = 1$ and $i = 3$, since this is the only way that the market-maker can buy the option for more than $\underline{C}$.

The problem to solve is to choose Δ and B such that

$$\Delta S_{2,1} + BR^2 = C_{2,1}$$

$$\Delta S_{2,3} + BR^2 = C_{2,3}.$$

In this simple two-period model, the problem is solved for a portfolio with $\Delta = \overline{\Delta}$ and $B = \overline{B}$, and the cost of the portfolio is $\overline{C}$, which was first introduced in Subsection 3.1. Constructing this portfolio gives the institution selling the option a profit $\Pi^b = -\overline{\Pi}$.

If the financial institution decides to buy the call option at the price of the hedging portfolio ($\overline{C}$) it will have a zero profit at time $t = 0$. However, since $\underline{C} < \overline{C}$, it will have an incentive to buy the option at a price C^b lower than $\overline{C}$ as long as $\underline{C} < C^b < \overline{C}$ is satisfied. Once again, profits at time $t = 2$ are a function only of the hedging portfolio. After deciding upon the hedging portfolio to construct, the financial institution will maximize its utility function choosing the selling price of the call option.

Again, it is relevant to understand the conditions under which the institution chooses to be fully hedged, buying the option at a price $C^b \leq \underline{C}$, or taking the chance of a negative profit at time $t = 2$. The expected utility under a fully hedged position is given by

$$E\mathcal{U}^{com}(\Pi_0, \Pi_2) = \underline{C} - C^b + p^2 v(0) + 2p(1-p)v(0) + (1-p)^2 v(\underline{\Pi}_{2,3}), \qquad (4)$$

On the other hand, when the financial institution buys the option at a price C^b such that $\underline{C} \leq C^b < \overline{C}$ and constructs the hedging portfolio $\left(\overline{\Delta}, \overline{B}\right)$, the expected utility is given by

$$E\mathcal{U}^{inc}(\Pi_0, \Pi_2) = \overline{C} - C^b + p^2 v(0) + 2p(1-p)v(-\overline{\Pi}_{2,2}) + (1-p)^2 v(0).$$

A sufficient condition for selling below $\underline{C}$ and being fully hedged is then easily found by imposing that

$$E\mathcal{U}^{com}(\Pi_0, \Pi_2, C^b = \underline{C}) \geq E\mathcal{U}^{inc}(\Pi_0, \Pi_2, C^b = \underline{C}). \tag{5}$$

Again, Since $E\mathcal{U}(\Pi_0, \Pi_2)$ is monotonic in C^b, this is also a necessary condition leading to

Proposition 4 *The financial institution chooses to buy the option at a price $C^b \leq \underline{C}$ if and only if*

$$\overline{C} - \underline{C} \leq (1-p)\left[-2pv(-\overline{\Pi}_{2,2}) + (1-p)v(\underline{\Pi}_{2,3}) - (1-3p)v(0)\right]. \tag{6}$$

Proof. The result follows[7] directly from inequality (5). ■

4.3 Illustration

In this work it is not possible to use the usual utility functions, such as power or logarithmic functions since portfolios may have non-positive terminal values. For simplicity, we use here the negative exponential type:

$$\mathcal{U} : x \to \lambda - \alpha \exp(-\delta x)$$

for given constants $\alpha, \lambda > 0$. The parameter δ is the constant coefficient of risk aversion. Fixing $\alpha = 1$ without loss of generality, the utility function of the financial institution $\mathcal{U}(\Pi_0, \Pi_2)$ is given by

$$E\mathcal{U}(\Pi_0, \Pi_2) = \Pi_0 + \sum_i \widehat{p}_i \left[\lambda - \exp\left(-\delta \Pi_{2,i}\right)\right],$$

[7] The case $R^2 < UD$ is studied in the appendix.

where $\hat{p}_i$ is the true probability of each state $i = 1, 2, 3$. In the simple two-period model, $\hat{p}_i = p^2$, $\hat{p}_i = 2p(1-p)$ and $\hat{p}_3 = (1-p)^2$.

Consider first the case $R^2 > UD$. For an institution selling options, a superreplicating portfolio is prefered if and only if condition (3) holds. Assuming the negative exponential utility function, this condition simplifies to

$$\frac{\overline{C} - \underline{C}}{(1-p)} + (1-3p) \leq \left[(1-p)\exp(\delta\underline{\Pi}_{2,3}) - 2p\exp(-\delta\overline{\Pi}_{2,2})\right]. \tag{7}$$

A similar preference for a superreplicating portfolio occurs when an institution is buying options and condition (6) holds. With the negative exponential utility function, this condition simplifies to

$$\frac{\overline{C} - \underline{C}}{(1-p)} + (3p-1) \leq \left[-(1-p)\exp(-\delta\underline{\Pi}_{2,3}) + 2p\exp\left(\delta\overline{\Pi}_{2,2}\right)\right], \tag{8}$$

Notice that $\underline{\Pi}_{2,3} \geq 0$ and $\overline{\Pi}_{2,2} \geq 0$. Let δ_a denote the unique value of δ that satisfies (7) as an equality and let δ_b denote the unique value of δ that satisfies (8) also as an equality. Then the following holds.

Proposition 5 *The minimum level of risk aversion that leads a financial institution with a negative exponential utility function to prefer simultaneously to sell options at price $C^a \geq \overline{C}$ and to buy options at price $C^b \leq \underline{C}$ is*

$$\delta^{\min} = \max\{\delta_a, \delta_b\}$$

Proof. Uniqueness of δ_a and δ_b follow trivially from the right-hand side of both conditions (7) and (8). Preference for selling above $\overline{C}$ means, therefore, that $\delta \geq \delta_a$. Similarly, preference for buying below $\underline{C}$ means that $\delta \geq \delta_b$. Finally, for both conditions to be satisfied simultaneously, we require that $\delta \geq \delta^{\min}$ as given in the statement and the result is proved. ■

When the risk-aversion coefficient increases, the right-hand sides of both conditions above increase. This means that the conditions are more easily satisfied for more-risk-averse than for low-risk-averse instituitions. In other words, the more-risk-averse the institution is, the more likely it is that it prefers to be fully hedged. In the limit where intermediaries are infinitely risk averse, the following result holds.

Proposition 6 *Infinitely risk-averse intermediaries will always choose to be fully hedged and the bid and ask prices for options are at or outside the bounds.*

Proof. In the limit $\delta \to \infty$, the right-hand side of conditions (7) and (8) diverge. Therefore, in this limit the intermediary will prefer to be fully hedged. ■

5 Utility Maximization

The problem facing the financial institution is now divided into two stages. First, the institution must choose the level of hedging or, equivalently, the domain on which the transaction price should be established. After deciding on the price region, the institution determines the price by maximizing its utility function. Since the variable price is found only in the first argument of the utility function, the one that refers to $t = 0$, this problem looks relatively simple.

For a given hedging portfolio, the institution maximizes utility by choosing the highest price in the allowed range of values. In particular, this means that if conditions (3) and (6) hold, the financial institution maximizes its utility by choosing an infinite price when selling, and a zero price when buying. Of course, there would be no one interested in transacting at these price values. The point is that we did not consider the demand and supply of call options that institutions face in the market.[8]

[8] Or that we assumed that the demand and supply curves were infinitely elastic.

In this section we introduce the demand and supply of options that financial institutions face. The next section studies the problem of a monopolist institution. In this situation the institution knows the demand and supply curves exactly, as well as the impact of prices on quantities demanded and supplied.

5.1 The demand curve

Assume that investors buy call options with two periods to maturity. Each investor is exogenously endowed with a vector of resources $e^a = \left(e_0^a, e_{2,1}^a, e_{2,2}^a, e_{2,3}^a\right)$ where e_0^a denotes the endowment at time $t = 0$ and $e_{2,i}^a, i = 1, 2, 3$ denotes the endowment at time $t = 2$ for the three states of nature. Investors buy call options at price C^a and at time $t = 2$, the options' payoff is simply given by $C_{2,i} = \max(0, S_{2,i} - K)$, for $i = 1, 2, 3$. Let Q^d denote the number of options purchased by an investor.

At date $t = 0$, the investor decides upon the number of units of the security to buy and his wealth is given by $W_0^a = e_0^a - Q^d C^a$. Then, at time $t = 2$, and in each state of nature $i = 1, 2, 3$, the investor will have wealth $W_{2,i}^a = e_{2,i}^a + Q^d C_{2,i}$.

Assuming that investors are risk averse, the simplest class of utility functions that can be considered is the so-called additively-separable utility functions

$$F(W_0^a, W_2^a) = W_0^a + \sum_{i=1}^{3} \widehat{p}_i f(W_2^a)$$

where f is continuous, twice differentiable, strictly increasing, and strictly concave, and $\widehat{p}_i$ is, as before, the true probability of each state of nature[9]. Each agent maximizes his utility function on Q^a, believing that he can buy as many options as he wants without affecting the option price.

[9]The vector $\widehat{p}$ can either represent a vector of commonly agreed upon probabilities or a vector of subjective probabilities, in which case it should be indexed by the investor. It is here assumed that $\widehat{p}$ is the same among investors, but results do not depend on this assumption.

One must also consider that investors can trade also in the underlying asset market[10]. The point is that investors can also construct the portfolio $\left(\overline{\Delta}, \overline{B}\right)$ that superreplicates the payoff value at maturity. It is expected that investors will not be willing to pay more than $\overline{C}$ in order to buy an option. Then, the demand of call options is given by the solution to

$$\begin{array}{l} \max\limits_{Q^d} e_0^a - Q^d C + \sum_i \hat{p}_i f\left(e_{2,i}^a + Q^d C_{2,i}\right) \\ \text{s.t. } C \leq \overline{C}. \end{array}$$

The first order condition, which in this case is necessary and sufficient for an optimum, implies that the amount Q^d of call options demanded satisfies

$$\begin{aligned} C^d(Q^d) &= \sum_i \hat{p}_i C_{2,i} f'(e_{2,i}^a + Q^d C_{2,i}) \text{ when } C < \overline{C} \\ Q^d &= 0 \text{ when } C \geq \overline{C}. \end{aligned}$$

One must now construct the aggregate market demand of call options. The aggregate demand is defined as a suitable sum of the demands arising from all investors. This work considers only one market, the market for call options with the properties described above. In this market, each investor is a price taker, that is, each investor takes the price as given and thinks that the quantity that he wishes to transact will not have an impact on market prices. In such a simple case, the aggregate demand is given by the aggregation of each individual demand curve. Therefore, the market demand will be zero for call option prices $C^a > \overline{C}$, and will be the horizontal sum of the demand curves of individual consumers.

[10]Here it is assumed that the cost of constructing the superreplicating portfolio is equal to all agents in the economy. This is a simplistic assumption, since there is some literature considering that investors may incur higher costs that financial intermediaries.

5.2 The supply curve

The supply side of the market is obtained in an analogous way. It is also considered that investors are risk averse. The main difference between buyers and sellers comes from their exogenous endowment. It is now assumed that each investor is exogenously endowed with a vector of resources $e^b = \left(e_0^b, e_{2,1}^b, e_{2,2}^b, e_{2,3}^b\right)$, where e_0^b denotes the endowment at time $t = 0$ and $e_{2,i}^b, i = 1, 2, 3$ denotes the endowment at time $t = 2$ for the three states of nature. Investors sell call options at price C^b at time $t = 0$. Let Q^s denote the number of units of options sold by an investor.

At date $t = 0$, the investor decides upon the number of units of the security to sell and his wealth is given by $W_0^b = e_0^b + Q^s C^b$. Then, at time $t = 2$, and in each state of nature $i = 1, 2, 3$, the investor will have wealth $W_{2,i}^b$.

As before, one must also consider that investors can trade in the underlying asset market. Therefore, investors are able to construct the portfolio $(\underline{\Delta}, \underline{B})$ that super-replicates the payoff value at maturity. It is expected that investors will not be willing to sell the option for less than $\underline{C}$. Then, the supply of call options is given by the solution to

$$\begin{array}{l} \max\limits_{Q^s} e_0^b + Q^s C^b + \sum_i \hat{p}_i f\left(e_{2,i}^b - Q^s C_{2,i}\right) \\ \text{s.t. } C \geq \underline{C}. \end{array}$$

The first order condition, which in this case is necessary and sufficient for an optimum, implies that the supplied amount Q^s of call options must satisfy

$$\begin{aligned} C^s(Q^s) &= \sum_i \hat{p}_i C_{2,i} f'(e_{2,i}^b - Q^s C_{2,i}) \text{ when } C^b > \underline{C} \\ Q^s &= 0 \text{ when } C^b \leq \underline{C}. \end{aligned}$$

As in the case of the market demand, the market supply is simply given by the horizontal sum of each individual supply curve. Then, the market supply is positively sloped for bid prices higher than $\underline{C}$, and zero otherwise. Figure 2 represents a generic demand curve and supply curve of options faced by financial intermediaries. We define the limiting minimum amount of options the intermediary can buy at price $C^b > \underline{C}$,

$$Q^{s'} = [C^s]^{-1} \left(\underline{C}\right),$$

and the minimum amount of options the intermediary can sell at price $C^a < \overline{C}$,

$$Q^{d'} = [C^s]^{-1} \left(\overline{C}\right).$$

6 A Monopolist Market-Maker

The monopolist market-maker's problem consists of choosing the bid and ask prices to maximize utility. The problem can also be solved choosing the optimal quantities to transact and the optimal hedging strategies. We assume that the market-maker must satisfy all market demand and supply at the ask and bid prices he sets. Let Q^a denote the quantity of options to sell, Q^b the quantity to buy, Q^{inc} the quantity to hedge incompletely and Q^{com} the quantity to hedge completely. The problem of the market-maker is

$$\begin{aligned}
\max_{\left\{Q^a, Q^b, Q^{inc}, Q^{com}\right\}} EU &= C^d(Q^a)Q^a - C^s(Q^b)Q^b \\
&\quad + C_{1j}Q^{inc} + \sum_i \widehat{p}_i v(Q^{inc}\alpha_{1j}) + C_{2j}Q^{com} + \sum_i \widehat{p}_i v(Q^{com}\alpha_{2j}) \\
s.t.\ Q^{inc} + Q^{com} &= \left(Q^a - Q^b\right) j + \left(Q^b - Q^a\right)(1 - j) \\
Q^{inc} &\geq 0, Q^{com} \geq 0, Q^a \geq 0, Q^b \geq 0
\end{aligned}$$

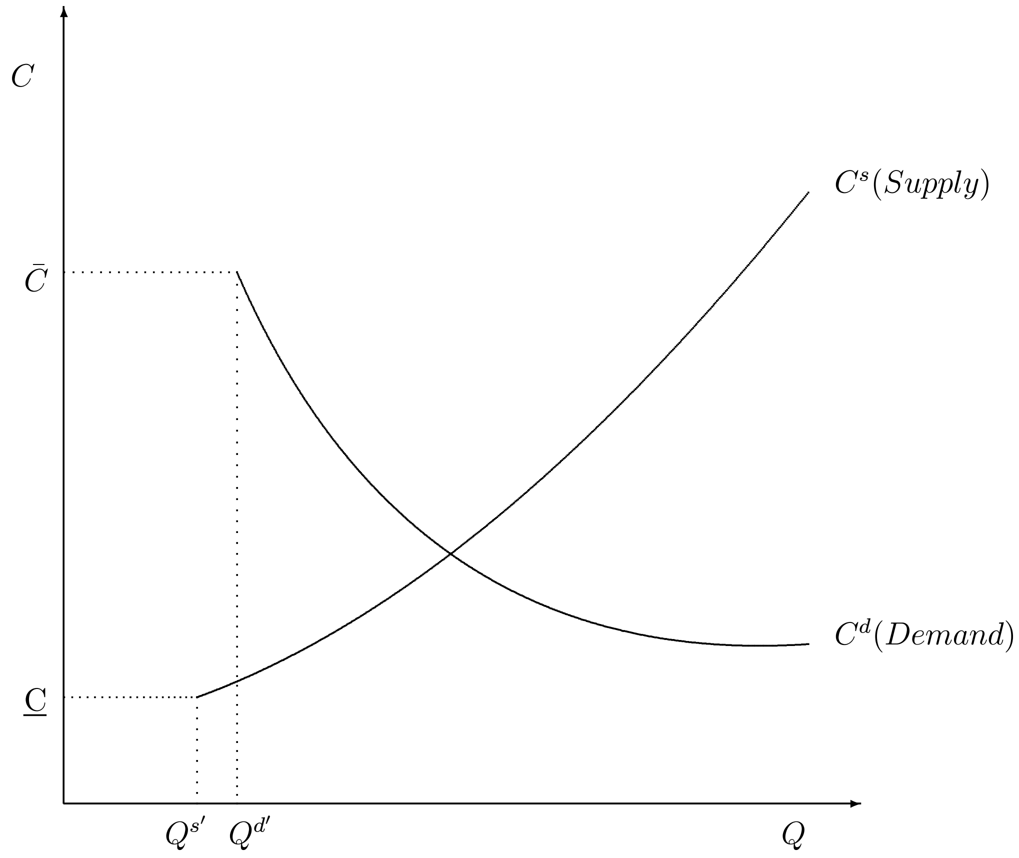

Figure 2: Market demand and supply of call options.

where

$$C_{1j} = \begin{cases} \overline{C},\ j=0 \\ -\underline{C},\ j=1 \end{cases} \quad C_{2j} = \begin{cases} \underline{C},\ j=0 \\ -\overline{C},\ j=1 \end{cases}$$

$$\alpha_{1j} = \begin{cases} -\overline{\Pi}_{2,i},\ j=0 \\ -\underline{\Pi}_{2,i},\ j=1 \end{cases} \quad \alpha_{2j} = \begin{cases} \underline{\Pi}_{2,i},\ j=0 \\ \overline{\Pi}_{2,i},\ j=1 \end{cases}$$

$$j = \begin{cases} 0 \text{ when } Q^b > Q^a \\ 1 \text{ when } Q^a > Q^b. \end{cases}$$

The utility function $v(\Pi_2)$, where Π_2 is the profit at maturity, is continuos and differentiable with $v'(\Pi_2) > 0$ and $v''(\Pi_2) < 0$. This characterizes a risk-averse market-maker. We assume that $v(\Pi_2) > 0$ for $\Pi_2 > 0$ and $v(\Pi_2) < 0$ for $\Pi_2 < 0$. The total revenues of selling Q^a options and total costs of buying Q^b options are respectively denoted by

$$\begin{aligned} TR^d &= C^d(Q^a)Q^a \\ TC^s &= C^s(Q^b)Q^b. \end{aligned}$$

When $j = 1$ the market-maker is selling more options that he is buying and $Q^a = Q^b + Q^{inc} + Q^{com}$. The total cost of the incomplete hedging portfolio to hedge Q^{inc} options and the total cost of the complete hedging portfolio to hedge Q^{com} are respectively given by

$$\begin{aligned} TC^{inc}\left(Q^{inc}\right) &= \underline{C}Q^{inc} - \sum_i \widehat{p}_i v(-Q^{inc}\underline{\Pi}_{2,i}), \\ TC^{com} &= \overline{C}Q^{com} - \sum_i \widehat{p}_i v(Q^{com}\overline{\Pi}_{2,i}). \end{aligned}$$

When $j = 0$ the market-maker is buying more options that he is selling and $Q^b = Q^a + Q^{inc} + Q^{com}$. The total revenue of the incomplete hedging portfolio to hedge Q^{inc} options and the total revenue of the complete hedging portfolio to hedge Q^{com} are respectively

$$\begin{aligned} TR^{inc} &= \overline{C}Q^{inc} + \sum_i \widehat{p}_i v(-\overline{\Pi}_{2,i}Q^{inc}), \\ TR^{com} &= \underline{C}Q^{com} + \sum_i \widehat{p}_i v(Q^{com}\underline{\Pi}_{2,i}). \end{aligned}$$

To solve the maximization problem it is necessary that all functions be diferentiable. However, $\frac{\partial TR^d}{\partial Q^a}$ is defined only for $Q^a \geq Q^{d'}$ and $\frac{\partial TC^s}{\partial Q^b}$ is defined only for $Q^b \geq Q^{s'}$.

On the other hand, we verify that

$$TC^{inc}(Q^{s'}) \;>\; TC^{s}(Q^{s'}) \text{ and}$$
$$TR^{d}(Q^{d'}) \;>\; TC^{inc}(Q^{d'}).$$

The optimal decision of the market-maker is to buy at least $Q^{s'}$ options and to sell $Q^{d'}$ options. Hence, we solve the maximization problem restricting $Q^a \geq Q^{d'}$ and $Q^b \geq Q^{s'}$. We are assuming that optimal transacted quantities of options are equal or higher than $\min\left\{Q^{d'}, Q^{s'}\right\}$.

The solution of the maximization problem is obtained for $\left\{Q^a, Q^b, Q^{inc}, Q^{com}\right\}$ such that

$$\frac{C^d(Q^a)+Q^a\frac{\partial C^d}{\partial Q^a}}{(2j-1)} = \frac{C^s(Q^b)+Q^b\frac{\partial C^s}{\partial Q^b}}{(2j-1)} =$$
$$-\left(C_{1j}+\textstyle\sum_i \widehat{p}_i\alpha_{1j}v'(Q^{inc}\alpha_{1j})\right) = -\left(C_{2j}+\textstyle\sum_i \widehat{p}_i\alpha_{2j}v'(Q^{com}\alpha_{2j})\right) \tag{9}$$
$$Q^a > Q^{d'},\; Q^b > Q^{s'},\; Q^{inc} > 0 \text{ and } Q^{com} > 0$$

When $j = 0$ condition (9) simplifies to

$$C^d(Q^a) + Q^a\frac{\partial C^d}{\partial Q^a} \;=\; C^s(Q^b) + Q^b\frac{\partial C^s}{\partial Q^b} = \tag{10}$$
$$\overline{C} - \sum_i \widehat{p}_i\overline{\Pi}_{2,i}v'(-\overline{\Pi}_{2,i}Q^{inc}) \;=\; \underline{C} + \sum_i \widehat{p}_i\underline{\Pi}_{2,i}v'(Q^{com}\underline{\Pi}_{2,i}) \tag{11}$$

which may be written as

$$MR^d(Q^a) = MC^s(Q^b) = MR^{inc}(Q^{inc}) = MR^{com}(Q^{com}).$$

When $j = 1$, condition (9) simplifies to

$$C^d(Q^a) + Q^a\frac{\partial C^d}{\partial Q^a} \;=\; C^s(Q^b) + Q^b\frac{\partial C^s}{\partial Q^b} = \tag{12}$$
$$\underline{C} + \sum_i \widehat{p}_i\underline{\Pi}_{2,i}v'(-\underline{\Pi}_{2,i}Q^{inc}) \;=\; \overline{C} - \sum_i \widehat{p}_i\overline{\Pi}_{2,i}v'(Q^{com}\overline{\Pi}_{2,i}) \tag{13}$$

and may be written as

$$MR^d(Q^a) = MC^s(Q^b) = MC^{inc}(Q^{inc}) = MC^{com}(Q^{com}).$$

The market-maker is maximizing utility when marginal revenues equal marginal costs. For example, when $Q^a > Q^b$ the market-maker is selling more options that he is buying. The difference $Q^a - Q^b$ in options is hedged constructing Q^{inc} incomplete hedging portfolios and Q^{com} complete hedging portfolios. The marginal cost of Q^b, Q^{inc} and Q^{com} must equal. Otherwise the market-maker could lower costs by switching quantities between the hedging alternatives. They must also equal the marginal revenue of sellling Q^a options.

It is shown in the appendix that

1. Conditions (10) and (12) cannot be verified simultaneously. If the optimal decision of the market-maker requires the construction of complete and incomplete hedging portfolios then he must be either a net seller or a net buyer of options.

2. The market maker may be maximizing utility without constructing complete hedging portfolios if and only if

$$\begin{aligned} MC^{com}(0) &\geq MR^d(Q^a) = MC^s(Q^b) = MC^{inc}(Q^{inc}), \text{ when } j = 1 \text{ or} \\ MR^{com}(0) &\leq MR^d(Q^a) = MC^s(Q^b) = MR^{inc}(Q^{inc}), \text{ when } j = 0. \end{aligned}$$

In this situation, the intermediary must decide between being a net seller or a net buyer. We will see in the illustration that follows how these conditions depend on the level of risk-aversion of the market-maker. In the appendix we present a graphical example of the determination of bid and ask prices when $Q^{com} = 0$ for both $j = 0$ and $j = 1$.

3. It is always optimal for the intermediary to buy and sell a different quantity. It means that the traditional solution of a monopolist firm characterized by

$$MR^d(Q) = MC^s(Q)$$

is never optimal.

6.1 Illustrations

As in Subsection 4.3, one can study the specific case of a constant risk-averse utility function. We use the negative exponential type utility function: $U : x \to \lambda - \exp(-\delta x)$ where λ is a positive constant taken as equal to one and δ is the constant coefficient of risk aversion. For this utility function the marginal costs of construction of a complete and an incomplete hedging portfolio are respectively given by

$$\begin{aligned} MC^{comp} &= \overline{C} - \Pi_{2,2}\widehat{p}_2\delta\exp(-\delta Q\Pi_{2,2}), \\ MC^{inc} &= \underline{C} + \underline{\Pi}_{2,3}\widehat{p}_3\delta\exp(\delta Q\underline{\Pi}_{2,3}), \end{aligned}$$

and the marginal revenues of construction of a complete and an incomplete hedging portfolio are respectively given by

$$\begin{aligned} MR^{comp} &= \underline{C} + \underline{\Pi}_{2,3}\widehat{p}_3\delta\exp(-\delta Q\underline{\Pi}_{2,3}), \\ MR^{inc} &= \overline{C} - \Pi_{2,2}\widehat{p}_2\delta\exp(\delta Q\Pi_{2,2}). \end{aligned}$$

The market-maker does not construct incomplete hedging portfolios, when $j = 0$, if

$$MR^{com}(0) \leq MR^{inc}(Q^{inc}) \Rightarrow \underline{C} + \underline{\Pi}_{2,3}\widehat{p}_3\delta \leq \overline{C} - \Pi_{2,2}\widehat{p}_2\delta\exp(\delta Q\Pi_{2,2}).$$

Let $\overline{Q}$ satisfiy the condition $\underline{C} + \underline{\Pi}_{2,3}\widehat{p}_3\delta - \overline{C} + \Pi_{2,2}\widehat{p}_2\delta\exp(\delta Q\Pi_{2,2}) = 0$. It is easy to see that $\frac{\partial\overline{Q}}{\partial\delta} < 0$. On the other hand, when $j = 1$ the market-maker does not construct incomplete hedging if

$$MC^{com}(0) \geq MC^{inc}(Q^{inc}) \Rightarrow \overline{C} - \Pi_{2,2}\widehat{p}_2\delta \geq \underline{C} + \underline{\Pi}_{2,3}\widehat{p}_3\delta \exp(\delta Q\underline{\Pi}_{2,3}).$$

Let $\widehat{Q}$ satisfy $\overline{C} - \Pi_{2,2}\widehat{p}_2\delta - \underline{C} - \underline{\Pi}_{2,3}\widehat{p}_3\delta \exp(\delta Q\underline{\Pi}_{2,3}) = 0$. It follows $\frac{\partial \widehat{Q}}{\partial \delta} < 0$. We can conclude that as δ increases, the intermediary moves from incomplete to complete hedging for a smaller quantity of options to hedge.

We can study the effect of a small variation of delta, all other things being equal, in the bid and ask prices. Consider the situation where the market-maker is selling more options than buying, and constructing an incomplete hedging portfolio. Optimal quantities must satisfy the following conditions

$$\begin{cases} MC^{inc}\left(Q^{inc}\right) - MC^{s}\left(Q^{b}\right) = 0 \\ Q^{inc} + Q^{b} = Q^{a} \\ MC^{s}\left(Q^{b}\right) - MR^{d}(Q^{a}) = 0. \end{cases}$$

The variation in the ask price is given by $\frac{\partial C^a(Q^a)}{\partial \delta} = \frac{\partial C^a(Q^a)}{\partial Q^a}\frac{\partial Q^a}{\partial \delta}$. The first term is the slope of the demand curve and is negative; the second term is shown in the appendix to be negative. Then, $\frac{\partial C^a(Q^a)}{\partial \delta} > 0$. On the other hand, $\frac{\partial C^b(Q^b)}{\partial \delta} = \frac{\partial C^b(Q^b)}{\partial Q^b}\frac{\partial Q^b}{\partial \delta}$ which is also positive. Both equilibrium prices increase but we cannot conclude about the size of the spread, since it depends on the characteristics of both the demand and supply curves. The increase in the coefficient of risk aversion has an opposite effect on prices when $Q^b > Q^a$. In this case $\frac{\partial C^a(Q^a)}{\partial \delta} < 0$ and $\frac{\partial C^b(Q^b)}{\partial \delta} < 0$. It is clear that in both cases the number of options to hedge decreases as the market-maker is more risk averse.

Consider the case of an infinitely risk-averse market-maker. As $\lim\limits_{\delta\rightarrow\infty} MC^{inc} = \infty$ and $\lim\limits_{\delta\rightarrow\infty} MR^{inc} = -\infty$, an infinitely risk-averse market-maker would never choose to be partially hedged. On the other hand, $\lim_{\delta\to\infty} MC^{comp} = \overline{C}$ and $\lim_{\delta\to\infty} MR^{comp} = \underline{C}$. Then, the market-maker would sell options at $\overline{C}$ and would buy options at $\underline{C}$. However, this would happen only if the demand and supply curves were infinitely elastic at $\overline{C}$ and $\underline{C}$, respectively. When the demand and supply curves are not infinitely elastic the market-maker maximizes his profits by choosing to sell and buy the same quantity Q^* satisfying

$$C^s(Q^*) + Q^* \frac{\partial C^s(Q^*)}{\partial Q} = C^d(Q^*) + Q^* \frac{\partial C^d(Q^*)}{\partial Q}.$$

In this situation the market-maker buys and sells the same quantity of options and does not construct any hedging portfolio. We studied the case of an infinite risk-averse market-maker in Section 4.3 and, consistently with the results of this section, we concluded that such a market-maker would never construct an incomplete hedging portfolio. However, we also concluded that prices would be equal to the bounds, which is different from the result that we have obtained in this section. The point is that in Section 4.3 we were assuming that demand and supply curves were infinitely elastic.

Finally, the necessary and sufficient conditions to obtain both prices at the bounds are:

$$\begin{aligned} MR^d(Q^{d\prime}) &< \overline{C} - \widehat{p}_2\overline{\Pi}_{2,2}\delta\exp(\delta\Pi_{2,2}Q^{inc}) = \underline{C} + \widehat{p}_3\underline{\Pi}_{2,3}\delta\exp(-\delta Q^{com}\underline{\Pi}_{2,3}) < MC^s(Q^{s\prime}) \\ Q^{inc} + Q^{com} &= Q^{s\prime} - Q^{d\prime} \end{aligned} \tag{14}$$

$$\begin{aligned} MR^d(Q^{d\prime}) &< \underline{C} + \widehat{p}_3\underline{\Pi}_{2,3}\delta\exp(\delta Q^{inc}\underline{\Pi}_{2,3}) = \overline{C} - \widehat{p}_2\overline{\Pi}_{2,2}\delta\exp(-\delta Q^{com}\underline{\Pi}_{2,2}) < MC^s(Q^{s\prime}) \\ Q^{inc} + Q^{com} &= Q^{d\prime} - Q^{s\prime}. \end{aligned} \tag{15}$$

In both cases it must be true that

$$C^d(Q^{d\prime}) + Q^{d\prime}\frac{\partial C^d(Q^{d\prime})}{\partial Q} < C^s(Q^{s\prime}) + Q^{s\prime}\frac{\partial C^s(Q^{s\prime})}{\partial Q}. \tag{16}$$

Notice that $C^s(Q) + Q\frac{\partial C^s}{\partial Q} = C^s(Q)\left[1 + \frac{1}{\varepsilon(s)}\right]$ where $\varepsilon(s) = \frac{\partial Q/Q}{\partial C^s/C^s}$ is the supply elasticity and $C^d(Q) + Q\frac{\partial C^d}{\partial Q} = C^d(Q)\left[1 - \frac{1}{|\varepsilon(d)|}\right]$ where $\varepsilon(d) = \frac{\partial Q/Q}{\partial C^d/C^d}$ is the demand elasticity. Then, conditions (16), (14) and (15) are more likely to be satisfied as the elasticity of both curves decreases. Conditions (14) and (15) also limit the values that δ may assume. These are necessary and sufficient conditions to obtain the equilibrium prices $C^a[\varepsilon(d), \varepsilon(s), \delta] = \overline{C}$ and $C^b[\varepsilon(d), \varepsilon(s), \delta] = \underline{C}$. In figure 3 we present an example of a situation where prices equal bounds. The monopolist market-maker buys $Q^{s'}$ options in the market at price $\underline{C}$ and sells $Q^{d'}$ options at price $\overline{C}$.

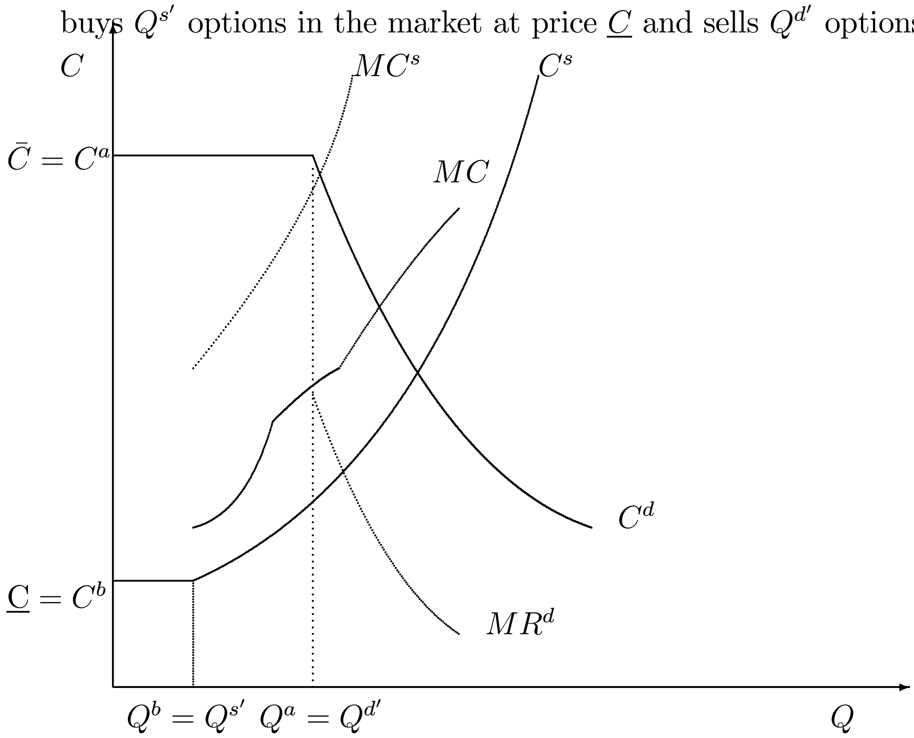

Figure 3: The monopoly case with prices equal to bounds

7 Conclusions

This paper studies the impact on option pricing of the impossibility of rebalancing the hedging portfolio. Since it is no longer possible to perfectly replicate the payoffs of an option, the traditional equilibrium valuation is no longer applicable. In this work, discrete rebalancing is not a result of transactions costs or wealth restrictions. The difficulty in rebalancing the hedging portfolio is a consequence of the reduced transaction in the underlying asset. Following Longstaff (2000), this expresses the asset's illiquidity.

First, we have shown that there is a range of non-arbitrage prices. This range is bounded from above by the value of the cheapest portfolio superreplicating a short position in the option. The lower bound corresponds to the value of the cheapest portfolio superreplicating a long position.

However, these bounds do not reflect equilibrium bid and ask prices for the options. An intermediary may be willing to accept the risk of having a negative profit at maturity and transact options at prices within the bounds. When market-makers do not cover all possible states of nature, the cost of the hedging portfolio is lower. It follows that market-makers may be willing to transact options at prices inside the bounds. To develop this analysis, we introduced market-makers that maximize their utilities, which are functions of the present and future profits.

We study a specific monopolistic market structure. The monopolist market-maker faces the market demand and supply curves for options and decides upon the bid and ask prices. He can hedge his position by transacting options in the market or by constructing a hedging portfolio. Moreover, the hedging portfolios may cover his position completely or incompletely. A utility maximizer market-maker will choose the cheapest among the three alternatives.

The impact of risk aversion on the pricing of options is analyzed for the specific case of a negative exponential utility function. We derive necessary conditions to observe the ask and bid prices at the bounds. These necessary conditions are related to the elasticity of the market demand and supply and the risk aversion of the market-maker. Prices are more likely to be at the bounds the less elastic the market curves are. We also show that as risk-aversion increases the rational decision of the market-maker is to move from an incomplete hedging strategy to a complete hedging strategy. In the limit, when the market-maker is infinitely risk averse, we show that he would never construct a partial hedging portfolio. In fact, as the cost of complete hedging turns out to be higher than transacting options in the market, an infinite risk-averse market-maker does not have open positions, that is, he buys and sells exactly the same quantity of options in the market.

In conclusion, in this monopolistic model for the market of options, the illiquidity of the underlying asset makes the equilibrium bid and ask price dependent on (1) the market-maker's risk aversion and (2) the elasticity of the demand and supply curves for options.

A Appendix

A.1 Proof of Proposition 1

The problem is given by $\min \Delta S + B$, choosing $\{\Delta, B\}$ subject to the terminal conditions: $\Delta S_{2,i} + BR^2 \geq C_{2,i}$ for $i = 1, 2, 3$. The solution follows from the Lagrangian:

$$\mathcal{L} = \Delta S + B + \sum_{i=1}^{3} \lambda_i (C_{2,i} - \Delta S_{2,i} - BR^2)$$

Alternatively, the problem can be solved following El Karoui and Quenez (1991) and (1995). To value a contingent claim in an incomplete market, these authors introduce the notion of auxiliary complete markets. In our case, each auxiliary market is characterized by only two active restrictions out of the three given above. Thus, there are three auxiliary complete markets in this problem. Each auxiliary market must satisfy the illiquidity restriction that no trade is possible at $t = 1$ and the additional restriction that the terminal wealth always exceeds or equals the payoff of the call option at maturity. It then follows that the value of the portfolio in the constrained market is given by the supremum of all portfolio values resulting from the auxiliary (complete) markets. In this two-period model it can be checked that only the auxiliary market resulting from $\lambda_1 > 0$, $\lambda_2 = 0$ and $\lambda_3 > 0$ satisfies the wealth restriction. The solution to the problem leads to a portfolio with $\overline{\Delta} = \frac{C_{2,1} - C_{2,3}}{S(U^2 - D^2)}$ and $\overline{B} = \frac{U^2 C_{2,3} - D^2 C_{2,1}}{R^2(U^2 - D^2)}$. The result follows.

A.2 Proof of Proposition 2

The problem is given by $\max \Delta S + B$, choosing $\{\Delta, B\}$ subject to the terminal conditions: $C_{2,i} \geq \Delta S_{2,i} + BR^2$ for $i = 1, 2, 3$. The solution follows from the Lagrangian:

$$\mathcal{L} = \Delta S + B + \sum_{i=1}^{3} \lambda_i (\Delta S_{2,i} + BR^2 - C_{2,i})$$

Once again, the problem is solved following Karatzas and Kou (1996). In this case, Karatzas and Kou's main result is that the value of the call option in the constrained market is given by the infimum of all call option prices resulting from the auxiliary (complete) markets. In this two-period model there are three auxiliary markets satisfying the illiquidity restriction, but it can be checked that only two of them satisfy the wealth condition, that is, that the value of the portfolio at maturity is always less than or equal to the payoff of the call option. These markets are the ones resulting from $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_3 = 0$ and $\lambda_1 = 0$, $\lambda_2 < 0$, $\lambda_3 < 0$. When $R^2 > UD$, the problem is solved with $\underline{\Delta} = \frac{C_{2,1} - C_{2,2}}{S(U^2 - UD)}$ and $\underline{B} = \frac{U^2 C_{2,2} - UDC_{2,1}}{R^2(U^2 - UD)}$. It follows that the cost of the hedging portfolio is the same as that given in the statement of this Proposition. When $UD > R^2$, the problem is solved with the different values of $\underline{\underline{\Delta}} = \frac{C_{2,2}-C_{2,3}}{S(UD-D^2)}$ and $\underline{\underline{B}} = \frac{UDC_{2,3}-D^2C_{2,2}}{R^2(UD-D^2)}$ and the cost of the portfolio is $\underline{\underline{C}} = \frac{1}{R^2}\left[\frac{R^2-D^2}{UD-D^2}C_{2,2} + \frac{UD-R^2}{UD-D^2}C_{2,3}\right]$.

A.3 Analysis of Section 4.1 when $R^2 < UD$.

When the hedging portfolio replicates only states $i = 2$ and $i = 3$ the portfolio is constructed with

$$\underline{\underline{\Delta}} = \frac{C_{2,2} - C_{2,3}}{S(UD - D^2)},$$
$$\underline{\underline{B}} = \frac{UDC_{2,3} - D^2C_{2,2}}{R^2(UD - D^2)}$$

and costs $\underline{\underline{C}} = \frac{1}{R^2}\left[\frac{R^2-D^2}{UD-D^2}C_{2,2} + \frac{UD-R^2}{UD-D^2}C_{2,3}\right]$. The resulting profit at time $t = 2$ is

$$\begin{aligned}
\underline{\underline{\Pi}}_{2,1} &= \frac{U}{D}(D^2S - K) < 0, \text{ if } UDS > K \geq SD^2 \\
&= K - U^2S < 0, \text{ if } U^2S > K \geq SDU \\
&= 0, \text{ otherwise} \\
\underline{\underline{\Pi}}_{2,2} &= 0 \\
\underline{\underline{\Pi}}_{2,3} &= 0
\end{aligned}$$

This situation is interesting to study when $R^2 < UD$ since it can be shown that, in such a case, $\underline{C} \leq \underline{\underline{C}}$. The point now is to understand the conditions under which the institution chooses to set prices above or below $\underline{\underline{C}}$. To do this, one must compute the expected utility obtained in the case of complete hedging

$$E\mathcal{U}^{com}(\Pi_0, \Pi_2) = \underline{\underline{C}} - C^b + p^2v(\underline{\underline{\Pi}}_{2,1}) + 2p(1-p)v(0) + (1-p)^2v(0).$$

and incomplete hedging

$$EU^{inc}(\Pi_0, \Pi_2) = \overline{C} - C^b + p^2v(0) + 2p(1-p)v(-\overline{\Pi}_{2,2}) + (1-p)^2v(0).$$

A sufficient and necessary condition for selling below $\underline{C}$ and being fully hedged is then easily found by imposing that

$$E\mathcal{U}^{com}(\Pi_0, \Pi_2, C^b = \underline{\underline{C}}) \geq E\mathcal{U}^{inc}(\Pi_0, \Pi_2, C^b = \underline{\underline{C}})$$

which simplifies to

$$\overline{C} - \underline{\underline{C}} \leq p^2 v(\underline{\underline{\Pi}}_{2,1}) + p(2-3p)v(0) - 2p(1-p)v(-\overline{\Pi}_{2,2}).$$

A.4 Proof of proposition 5 when $R^2 < UD$.

Proposition 5 is also valid for the case where $R^2 < UD$. In the presence of negative utility functions conditions (7) and (8) are now given by

$$\frac{\overline{C} - \underline{\underline{C}}}{p} + (3p-2) \leq \left[-2(1-p)\exp(-\delta_a \overline{\Pi}_{2,2}) + p\exp(\delta_a \underline{\underline{\Pi}}_{2,1})\right]$$

and

$$\frac{\overline{C} - \underline{\underline{C}}}{p} + (2-3p) \leq \left[2(1-p)\exp\left(\delta_b \Pi_{2,2}\right) - p\exp(-\delta_b \underline{\underline{\Pi}}_{2,1})\right].$$

One can determine the unique values for δ_a and δ_b satisfying these conditions. It follows the same proof as in the text.

A.5 The market-maker's problem

The maximization problem is

$$\begin{aligned}
\max_{\left\{Q^a, Q^b, Q^{inc}, Q^{com}\right\}} EU &= C^d(Q^a)Q^a - C^s(Q^b)Q^b \\
&\quad + C_{1j}Q^{inc} + \sum_i \widehat{p}_i v(Q^{inc}\alpha_{1j}) + C_{2j}Q^{com} + \sum_i \widehat{p}_i v(Q^{com}\alpha_{2j}) \\
s.t.\ Q^{inc} + Q^{com} &= \left(Q^a - Q^b\right) j + \left(Q^b - Q^a\right)(1-j) \\
Q^{inc} &\geq 0, Q^{com} \geq 0, Q^a \geq Q^{d'}, Q^b \geq Q^{s'}
\end{aligned}$$

The Lagrangean function is

$$\begin{aligned}
\mathcal{L} &= C^d(Q^a)Q^a - C^s(Q^b)Q^b \\
&\quad + C_{1j}Q^{inc} + \sum_i \widehat{p}_i v(Q^{inc}\alpha_{1j}) \\
&\quad + C_{2j}Q^{com} + \sum_i \widehat{p}_i v(Q^{com}\alpha_{2j}) \\
&\quad + \lambda\left(Q^{inc} + Q^{com} - \left(Q^a - Q^b\right) j - \left(Q^a - Q^b\right)(j-1)\right)
\end{aligned}$$

where λ is a positive Lagrange multiplier for $j = 1$ and negative for $j = 0$.

A.5.1 First Order Conditions

The first order conditions of the problem are

$$\begin{array}{lll}
MR^d(Q^a) - \lambda(2j-1) \leq 0 & Q^a \geq Q^{d\prime} & \left(MR^d(Q^{a\prime}) - \lambda(2j-1)\right)\left(Q^a - Q^{d\prime}\right) = 0 \\
-MC^s(Q^b) + \lambda(2j-1) \leq 0 & Q^b \geq Q^{s\prime} & \left(-MC^s(Q^{b\prime}) + \lambda(2j-1)\right)\left(Q^b - Q^{s\prime}\right) = 0 \\
C_{1j} + \sum_i \widehat{p}_i \alpha_{1j} v'(Q^{inc}\alpha_{1j}) + \lambda \leq 0 & Q^{inc} \geq 0 & \left(C_{1j} + \sum_i \widehat{p}_i \alpha_{1j} v'(Q^{inc}\alpha_{1j}) + \lambda\right) Q^{inc} = 0 \\
C_{2j} + \sum_i \widehat{p}_i \alpha_{2j} v'(Q^{com}\alpha_{2j}) + \lambda \leq 0 & Q^{com} \geq 0 & \left(C_{2j} + \sum_i \widehat{p}_i \alpha_{2j} v'(Q^{com}\alpha_{2j}) + \lambda\right) Q^{com} = 0
\end{array}$$

$$Q^{inc}+Q^{com}-\left(Q^{a}-Q^{b}\right)j-\left(Q^{b}-Q^{a}\right)(1-j)=0$$

Case 1 -

$Q^{a}-Q^{d\prime}>0,\quad Q^{b}-Q^{s\prime}>0,\quad Q^{inc}>0,\quad Q^{com}>0$. The first order conditions simplify to

$$\frac{MR^{d}(Q^{a})}{(2j-1)}=\frac{MC^{s}(Q^{b})}{(2j-1)}=-\left(C_{1j}+\sum_{i}\widehat{p}_{i}\alpha_{1j}v'(Q^{inc}\alpha_{1j})\right)=-\left(C_{2j}+\sum_{i}\widehat{p}_{i}\alpha_{2j}v'(Q^{com}\alpha_{2j})\right).$$

When $j=0$ it simplifies to

$$MR^{d}(Q^{a})=MC^{s}(Q^{b})=\overline{C}-\sum_{i}\widehat{p}_{i}\overline{\Pi}_{2,i}v'(-\overline{\Pi}_{2,i}Q^{inc})=\underline{C}+\sum_{i}\widehat{p}_{i}\underline{\Pi}_{2,i}v'(Q^{com}\underline{\Pi}_{2,i})$$

and when $j=1$

$$MR^{d}(Q^{a})=\lambda=MC^{s}(Q^{b})=\underline{C}+\sum_{i}\widehat{p}_{i}\underline{\Pi}_{2,i}v'(-\underline{\Pi}_{2,i}Q^{inc})=\overline{C}-\sum_{i}\widehat{p}_{i}\overline{\Pi}_{2,i}v'(Q^{com}\overline{\Pi}_{2,i}).$$

Notice that by the first order condition the optimal quantity

$$(Q^{a}\,|j=0)<(Q^{a}\,|j=1)$$

which implies that $MR^{d}\left(Q^{a}\,|j=0\right)>MR^{d}\left(Q^{a}\,|j=1\right)$ by monotonicity of $MR^{d}(Q)$. It follows from the first order conditions that

$$\left[\overline{C}-\sum_{i}\widehat{p}_{i}\overline{\Pi}_{2,i}v'(-\overline{\Pi}_{2,i}Q^{inc})\,|j=0\right]>\left[\overline{C}-\sum_{i}\widehat{p}_{i}\overline{\Pi}_{2,i}v'(Q^{com}\overline{\Pi}_{2,i})\,|j=1\right]\Rightarrow$$
$$\left[\overline{C}-\sum_{i}\widehat{p}_{i}\overline{\Pi}_{2,i}v'(0)\,|j=0\right]>\left[\overline{C}-\sum_{i}\widehat{p}_{i}\overline{\Pi}_{2,i}v'(0)\,|j=1\right]$$

which is false. Also by the first order conditions follows that

$$\left[\underline{C}+\sum_{i}\widehat{p}_{i}\underline{\Pi}_{2,i}v'(Q^{com}\underline{\Pi}_{2,i})\,|j=0\right]>\left[\underline{C}+\sum_{i}\widehat{p}_{i}\underline{\Pi}_{2,i}v'(-\underline{\Pi}_{2,i}Q^{inc})\,|j=1\right]\Rightarrow$$
$$\left[\underline{C}+\sum_{i}\widehat{p}_{i}\underline{\Pi}_{2,i}v'(0)\,|j=0\right]>\left[\underline{C}+\sum_{i}\widehat{p}_{i}\underline{\Pi}_{2,i}v'(0)\,|j=1\right]$$

which is false. We conclude that there is only one optimal solution either when $j = 0$ or when $j = 1$.

Case 2 -

$Q^a - Q^{d\prime} > 0, \quad Q^b - Q^{s\prime} > 0, \quad Q^{inc} > 0, \quad Q^{com} = 0$. The first order conditions simplify to

$$-\left(C_{2j} + \sum_i \widehat{p}_i \alpha_{2j} v'(0)\right) > \frac{MR^d(Q^a)}{(2j-1)} = \frac{MC^s(Q^b)}{(2j-1)} = -\left(C_{1j} + \sum_i \widehat{p}_i \alpha_{1j} v'(Q^{inc}\alpha_{1j})\right)$$

When $j = 0$ these conditions simplify to

$$MR^d(Q^a) = MC^s(Q^b) = \overline{C} - \sum_i \widehat{p}_i \overline{\Pi}_{2,i} v'(-\overline{\Pi}_{2,i} Q^{inc}) > \underline{C} + \sum_i \widehat{p}_i \underline{\Pi}_{2,i} v'(0)$$

and when $j = 1$ these conditions simplify

$$\overline{C} - \sum_i \widehat{p}_i \overline{\Pi}_{2,i} v'(0) > MR^d(Q^a) = MC^s(Q^b) = \underline{C} + \sum_i \widehat{p}_i \underline{\Pi}_{2,i} v'(-\underline{\Pi}_{2,i} Q^{inc}).$$

The market-maker chooses between being a net seller or a ner buyer of options.

Case 3 -

$Q^a - Q^{d\prime} > 0, \quad Q^b - Q^{s\prime} > 0, \quad Q^{inc} = 0, \quad Q^{com} = 0$. The first order conditions simplify to

$$\begin{aligned} -\left(C_{1j} + \sum_i \widehat{p}_i \alpha_{1j} v'(0)\right) &> \frac{MR^d(Q^a)}{(2j-1)} = \frac{MC^s(Q^b)}{(2j-1)} \\ -\left(C_{2j} + \sum_i \widehat{p}_i \alpha_{2j} v'(0)\right) &> \frac{MR^d(Q^a)}{(2j-1)} = \frac{MC^s(Q^b)}{(2j-1)} \end{aligned}$$

and $Q^a = Q^b = Q$. When $j = 0$ these conditions simplify to

$$\begin{aligned} \overline{C} - \sum_i \widehat{p}_i \overline{\Pi}_{2,i} v'(0) &= MR^{inc}(0) < MR^d(Q) = MC^s(Q), \qquad (17) \\ \underline{C} + \sum_i \widehat{p}_i \underline{\Pi}_{2,i} v'(0) &= MR^{com}(0) < MR^d(Q) = MC^s(Q), \end{aligned}$$

and when $j = 1$ these conditions simplify to

$$\begin{aligned}
\underline{C} + \sum_i \widehat{p}_i \underline{\Pi}_{2,i} v'(0) &= MC^{inc}(0) > MR^d(Q) = MC^s(Q), \\
\overline{C} - \sum_i \widehat{p}_i \overline{\Pi}_{2,i} v'(0) &= MC^{com}(0) > MR^d(Q) = MC^s(Q).
\end{aligned} \tag{18}$$

Notice that these conditions cannot be verified simultaneously. Suppose (17) is true. Then $MC^{inc}(0) = MR^{inc}(0) < MR^d(Q) = MC^s(Q)$ which contradicts (18). But if $MC^{inc}(0) < MR^d(Q) = MC^s(Q)$ it is not optimal to choose $Q^{inc} = 0$. On the other hand suppose (18) is true. Then, (17) is false and it is not optimal to have $Q^{inc} = 0$. It follows that it is never optimal to transact the same quantity on both sides of the spread.

Case 4 -

$Q^a - Q^{d\prime} > 0, \quad Q^b - Q^{s\prime} > 0, \quad Q^{inc} = 0, \quad Q^{com} > 0$. The first order conditions simplify to

$$-\left(C_{1j} + \sum_i \widehat{p}_i \alpha_{1j} v'(Q^{inc}\alpha_{1j})\right) > \frac{MR^d(Q^a)}{2j-1} = \frac{MC^s(Q^b)}{2j-1} = -\left(C_{2j} + \sum_i \widehat{p}_i \alpha_{2j} v'(Q^{com}\alpha_{2j})\right).$$

These conditions simplify when $j = 0$ to

$$\overline{C} - \sum_i \widehat{p}_i \overline{\Pi}_{2,i} v'(0) < MR^d(Q^a) = MC^s(Q^b) = \underline{C} + \sum_i \widehat{p}_i \underline{\Pi}_{2,i} v'(Q^{com}\underline{\Pi}_{2,i})$$

and when $j = 1$ simplify to

$$\underline{C} + \sum_i \widehat{p}_i \underline{\Pi}_{2,i} v'(0) > MR^d(Q^a) = MC^s(Q^b) = \overline{C} - \sum_i \widehat{p}_i \overline{\Pi}_{2,i} v'(Q^{com}\overline{\Pi}_{2,i}).$$

Case 5 -

$Q^a - Q^{d\prime} = 0, \quad Q^b - Q^{s\prime} = 0, \quad Q^{inc} > 0, \quad Q^{com} > 0$. The first order conditions simplify to

$$MR^d(Q^{d\prime}) < -\left(C_{1j} + \sum_i \widehat{p}_i \alpha_{1j} v'(Q^{inc}\alpha_{1j})\right)(2j-1) = -\left(C_{2j} + \sum_i \widehat{p}_i \alpha_{2j} v'(Q^{com}\alpha_{2j})\right)(2j-1) <$$

and $Q^{inc} + Q^{com} = \left(Q^{d\prime} - Q^{s\prime}\right) j + \left(Q^{d\prime} - Q^{s\prime}\right)(j-1)$. When $j = 0$

$$MR^d(Q^{d\prime}) < \overline{C} - \sum_i \widehat{p}_i \overline{\Pi}_{2,i} v'(-\overline{\Pi}_{2,i} Q^{inc}) = \underline{C} + \sum_i \widehat{p}_i \underline{\Pi}_{2,i} v'(Q^{com}\underline{\Pi}_{2,i}) < MC^s(Q^{s\prime})$$

and when $j = 1$

$$MR^d(Q^{d\prime}) < \underline{C} + \sum_i \widehat{p}_i \underline{\Pi}_{2,i} v'(-Q^{inc}\underline{\Pi}_{2,i}) = \overline{C} - \sum_i \widehat{p}_i \overline{\Pi}_{2,i} v'(Q^{com}\overline{\Pi}_{2,i}) < MC^s(Q^{s\prime}).$$

Case 6 -

$Q^a - Q^{d\prime} = 0, \quad Q^b - Q^{s\prime} = 0, \quad Q^{inc} > 0, \quad Q^{com} = 0$. The first order conditions simplify to

$$MR^d(Q^{d\prime}) < -\left(C_{1j} + \sum_i \widehat{p}_i \alpha_{1j} v'(Q^{inc}\alpha_{1j})\right)(2j-1) < MC^s(Q^{s\prime})$$

$$\left(C_{2j} + \sum_i \widehat{p}_i \alpha_{2j} v'(Q^{com}\alpha_{2j})\right) < \left(C_{1j} + \sum_i \widehat{p}_i \alpha_{1j} v'(Q^{inc}\alpha_{1j})\right)$$

and $Q^{inc} + Q^{com} = \left(Q^{d\prime} - Q^{s\prime}\right) j + \left(Q^{d\prime} - Q^{s\prime}\right)(j-1)$. When $j = 0$

$$\begin{aligned} MR^d(Q^{d\prime}) &< \overline{C} - \sum_i \widehat{p}_i \overline{\Pi}_{2,i} v'(-\overline{\Pi}_{2,i} Q^{inc}) < MC^s(Q^{s\prime}) \\ \left(\underline{C} + \sum_i \widehat{p}_i \underline{\Pi}_{2,i} v'(0)\right) &< \left(\overline{C} - \sum_i \widehat{p}_i \overline{\Pi}_{2,i} v'(Q^{inc}\overline{\Pi}_{2,i})\right) \end{aligned}$$

and when $j = 1$

$$\begin{aligned} MR^d(Q^{d\prime}) &< \underline{C} + \sum_i \widehat{p}_i \underline{\Pi}_{2,i} v'(-Q^{inc}\underline{\Pi}_{2,i}) < MC^s(Q^{s\prime}) \\ \overline{C} - \sum_i \widehat{p}_i \overline{\Pi}_{2,i} v'(0) &> \underline{C} + \sum_i \widehat{p}_i \underline{\Pi}_{2,i} v'(-\underline{\Pi}_{2,i} Q^{inc}). \end{aligned}$$

A.5.2 Second Order Conditions

The first order conditions are necessary and sufficient to the determination of the maximum when the objective function is concave. The second derivative of the objective function is given by

$$\begin{aligned}
\frac{\partial^2 EU}{\partial Q^2} &= \frac{\partial MR^d}{\partial Q} - \frac{\partial MC^s}{\partial Q} + \frac{\partial MR^{inc}}{\partial Q} + \frac{\partial MR^{com}}{\partial Q} \text{ when } j = 0, \\
\frac{\partial^2 EU}{\partial Q^2} &= \frac{\partial MR^d}{\partial Q} - \frac{\partial MC^s}{\partial Q} - \frac{\partial MC^{inc}}{\partial Q} - \frac{\partial MC^{com}}{\partial Q} \text{ when } j = 1.
\end{aligned}$$

Notice that $\frac{\partial MR^{inc}}{\partial Q} + \frac{\partial MR^{com}}{\partial Q} < 0$ and $-\frac{\partial MC^{inc}}{\partial Q} - \frac{\partial MC^{com}}{\partial Q} < 0$. Then, the sufficient conditions for $\frac{\partial^2 EU}{\partial Q^2} < 0$ is $\frac{\partial MR^d}{\partial Q} < 0$ and $\frac{\partial MC^s}{\partial Q} > 0$.

A.6 Graphical illustrations

Figure 4 characterizes a situation with $Q^{com} = 0$ and $Q^b < Q^a$. MC is the marginal cost curve of the market-maker and is given by the horizontal sum of each individual marginal cost curves. In this case the market-maker buys $Q^a - Q^b$ portfolios $(\underline{\Delta}, \underline{B})$ and has the expected utility given by

$$EU = C^a(Q^a)Q^a - C^b(Q^b)Q^b - \left[\underline{C}\left(Q^a - Q^b\right) - \sum_i \widehat{p}_i v(-\left(Q^a - Q^b\right)\underline{\Pi}_{2,i})\right].$$

Figure 5 characterizes a situation where $j = 0$ and $Q^{com} = 0$. Optimal quantities satisfy $Q^b > Q^a$. MR is the marginal revenue curve of the market-maker and is given by the horizontal sum of each individual marginal revenue curves. The market maker is selling $Q^b - Q^a$ portfolios $\left(\overline{\Delta}, \overline{B}\right)$ and has the expected utility

$$EU = C^a(Q^a)Q^a - C^b(Q^b)Q^b - \left[\overline{C}\left(Q^b - Q^a\right) - \sum_i \widehat{p}_i v(-\left(Q^b - Q^a\right)\overline{\Pi}_{2,i})\right].$$

The market-maker will choose the situation that gives him the highest expected utility. In both cases the equilibrium is characterized by

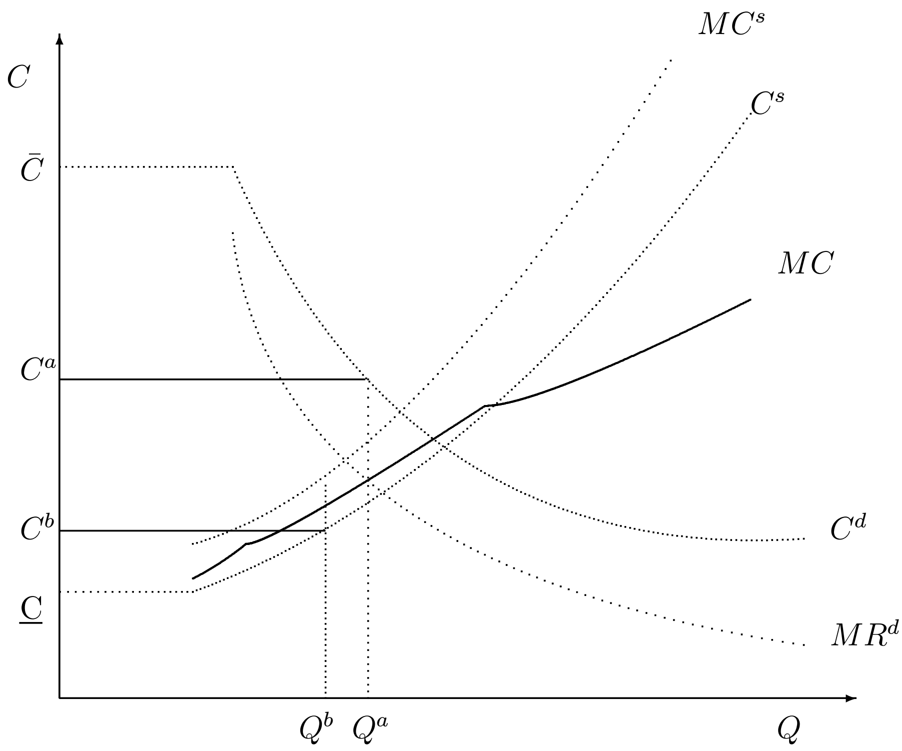

Figure 4: The bid and ask prices when $Q^a - Q^b > 0$.

- the ask price is lower than $\overline{C}$;
- the bid price is higher than $\underline{C}$;
- the quantity sold by the market-maker is different from the quantity bought;
- the market maker hedges his position buying the hedging portfolio $(\underline{\Delta}, \underline{B})$ or selling the portfolio $\left(\overline{\Delta}, \overline{B}\right)$.

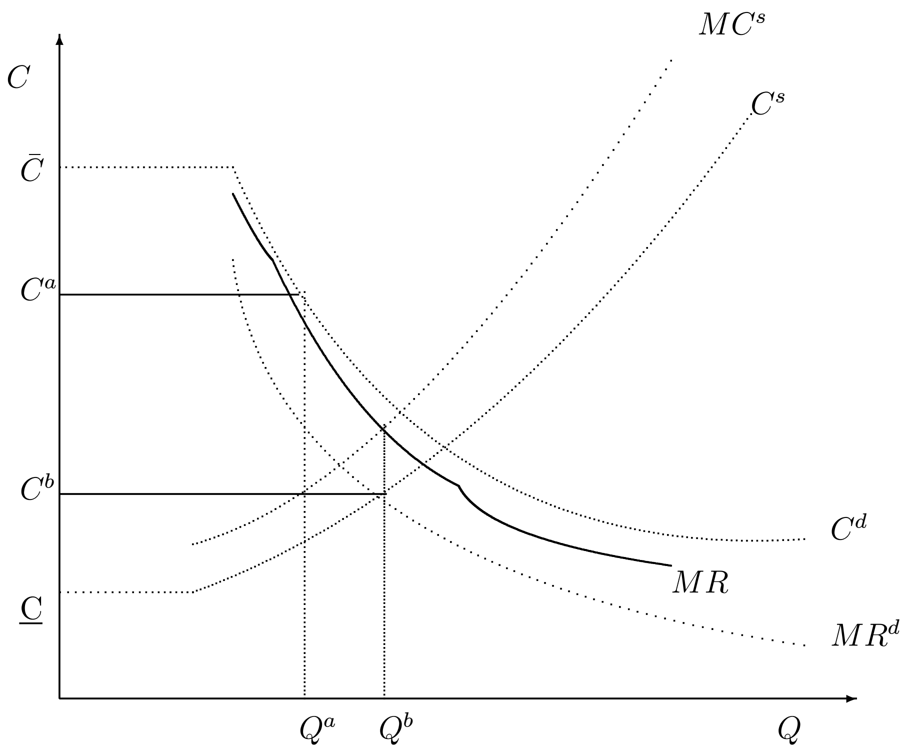

Figure 5: The bid and ask prices when $Q^a - Q^b < 0$.

A.7 Analysis of changes in prices when the coefficient of risk aversion changes.

When the market-maker is selling more options than he is buying and hedging incompletely, the optimal quantitites Q^a and Q^b satisfy the following conditions:

$$MC^{inc}\left(Q^{inc}\right) - MC^s\left(Q^b\right) = 0$$

$$Q^{inc} + Q^b = Q^a$$

$$MC^s\left(Q^b\right) - MR^d(Q^a) = 0$$

We want to compute the variations in Q^b and Q^a when the coefficient of risk aversion δ changes. Then, we must solve the following:

$$\begin{pmatrix} \frac{\partial}{\partial\delta}Q^{inc} \\ \frac{\partial}{\partial\delta}Q^b \\ \frac{\partial}{\partial\delta}Q^a \end{pmatrix} = -\begin{pmatrix} \frac{\partial MC^{inc}}{\partial Q^{inc}} & -\frac{\partial MC^s}{\partial Q^b} & 0 \\ 1 & 1 & -1 \\ 0 & \frac{\partial MC^s}{\partial Q^b} & -\frac{\partial MR^d}{\partial Q^s} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial MC^{inc}}{\partial\delta} \\ 0 \\ 0 \end{pmatrix}$$

$$= A \begin{pmatrix} -\frac{\partial MR^d}{\partial Q^s} + \frac{\partial MC^s}{\partial Q^b} \\ \frac{\partial MR^d}{\partial Q^s} \\ \frac{\partial MC^s}{\partial Q^b} \end{pmatrix}$$

where

$$A = \frac{\frac{\partial MC^{inc}}{\partial\delta}}{\frac{\partial MC^{inc}}{\partial Q^{inc}}\frac{\partial MR^d}{\partial Q^a} - \frac{\partial MC^{inc}}{\partial Q^{inc}}\frac{\partial MC^s}{\partial Q^b} + \frac{\partial MC^s}{\partial Q^b}\frac{\partial MR^d}{\partial Q^a}} < 0.$$

Then, $\frac{\partial}{\partial\delta}Q^{inc} < 0$; $\frac{\partial}{\partial\delta}Q^b > 0$ and $\frac{\partial}{\partial\delta}Q^a < 0$.

When the market-maker is buying more options than he is selling, the optimal quantitites Q^a and Q^b satisfy the following conditions:

$$MR^{inc}\left(Q^{inc}\right) - MR^d\left(Q^a\right) = 0$$

$$Q^{inc} + Q^a = Q^b$$

$$MC^s\left(Q^b\right) - MR^d(Q^a) = 0$$

We want to compute the variations in Q^b and Q^a when the coefficient of risk aversion δ changes.

$$\begin{pmatrix} \frac{\partial}{\partial\delta}Q^{inc} \\ \frac{\partial}{\partial\delta}Q^b \\ \frac{\partial}{\partial\delta}Q^a \end{pmatrix} = -\begin{pmatrix} \frac{\partial MR^{inc}}{\partial Q^{inc}} & 0 & -\frac{\partial MR^d}{\partial Q^a} \\ 1 & -1 & 1 \\ 0 & \frac{\partial MC^s}{\partial Q^b} & -\frac{\partial MR^d}{\partial Q^a} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial MR^{inc}}{\partial\delta} \\ 0 \\ 0 \end{pmatrix}$$

$$A \begin{pmatrix} -\frac{\partial MR^d}{\partial Q^a} + \frac{\partial MC^s}{\partial Q^b} \\ -\frac{\partial MR^d}{\partial Q^a} \\ -\frac{\partial MC^s}{\partial Q^b} \end{pmatrix}$$

where

$$A = \frac{-\frac{\partial MR^{inc}}{\partial\delta}}{-\frac{\partial MR^{inc}}{\partial Q^{inc}}\frac{\partial MR^d}{\partial Q^a} + \frac{\partial MR^{inc}}{\partial Q^{inc}}\frac{\partial MC^s}{\partial Q^b} + \frac{\partial MR^d}{\partial Q^a}\frac{\partial MC^s}{\partial Q^b}} < 0.$$

Then, $\frac{\partial}{\partial\delta}Q^{inc} < 0$; $\frac{\partial}{\partial\delta}Q^b < 0$ and $\frac{\partial}{\partial\delta}Q^a > 0$.

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